

KILLING AND TWISTOR SPINORS WITH TORSION

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ABSTRACT. We study twistor spinors (with torsion) on Riemannian spin manifolds (M^n, g, T) carrying metric connections with totally skew-symmetric torsion. We consider the characteristic connection $\nabla^c = \nabla^g + \frac{1}{2}T$ and under the condition $\nabla^c T = 0$, we show that the twistor equation with torsion w.r.t. the family $\nabla^s = \nabla^g + 2sT$ can be viewed as a parallelism condition under a suitable connection on the bundle $\Sigma \oplus \Sigma$, where Σ is the associated spinor bundle. Consequently, we prove that a twistor spinor with torsion has isolated zero points. Next we study a special class of twistor spinors with torsion, namely these which are T -eigenspinors and parallel under the characteristic connection; we show that the existence of such a spinor for some $s \neq 1/4$ implies that (M^n, g, T) is both Einstein and ∇^c -Einstein, in particular the equation $\text{Ric}^s = \frac{\text{Scal}^s}{n}g$ holds for any $s \in \mathbb{R}$. In fact, for ∇^c -parallel spinors we provide a correspondence between the Killing spinor equation with torsion and the Riemannian Killing spinor equation. This allows us to describe 1-parameter families of non-trivial Killing spinors with torsion on nearly Kähler manifolds and nearly parallel G_2 -manifolds, in dimensions 6 and 7, respectively, but also on the 3-dimensional sphere S^3 . We finally present applications related to the universal and twistorial eigenvalue estimate of the square of the cubic Dirac operator.

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INTRODUCTION

Consider a connected Riemannian spin manifold (M^n, g) carrying a family of metric connections ∇^s with (totally) skew-symmetric torsion, say $4sT$ for some non-trivial 3-form $T \in \bigwedge^3 M$ and $s \in \mathbb{R}$. We use the same symbol for the lift of ∇ on the spinor bundle $\Sigma \rightarrow M$ and denote by $X \cdot \varphi := \mu(X \otimes \varphi)$ the Clifford multiplication $\mu : TM \otimes \Sigma \rightarrow \Sigma$ at the bundle level. If there exist some $s \neq 0$ and $\zeta \neq 0$ such that

$$\nabla_X^s \varphi = \zeta X \cdot \varphi, \quad (*)$$

for any vector field X on M , then the spinor field $\varphi \in \Gamma(\Sigma)$ is called a *Killing spinor with torsion* (KsT in short). For $s = 0$ (zero torsion), this equation reduces to the Riemannian Killing spinor equation with its strong geometric consequences, see for example [15, 28]. Similarly, a *twistor spinor with torsion* (TsT in short), is a section $\varphi \in \Gamma(\Sigma)$ satisfying the twistor equation with respect to ∇^s for some $s \neq 0$, i.e.

$$\nabla_X^s \varphi + \frac{1}{n} X \cdot D^s \varphi = 0, \quad (**)$$

where $D^s = \mu \circ \nabla^s$ is the induced Dirac operator. The interest in TsT and in some special cases KsT (e.g. in dimension 6, see [3], but also in dimension 7 as we will show in the present work), is due to the fact that they realize the equality case for eigenvalue estimates of the cubic Dirac operator $\not{D} = D^g + \frac{1}{4}T$, i.e. the Dirac operator associated to the connection with torsion $T/3$. These estimates occur both in the presence of parallel torsion. The so-called *universal estimate* β_{univ} is based on a generalized formula of Schrödinger-Lichnerowicz type, see [16, 41, 24, 31, 1, 6] for a description of the long history related with the square of \not{D} . The *twistorial estimate* β_{tw} is more recent and relies on a method using Penrose's twistor operator associated to ∇^s , see [3, 17]. The equality case for this estimate takes place when the Riemannian scalar curvature is constant and the spinor is a twistor spinor with torsion (TsT) for a specific parameter s (depending on the dimension n of M). Hence, it gives rise to a technique available for the construction of non-trivial TsT. It is an interesting question to check if these twistor spinors with torsion are also some kind of Killing spinors and what the geometric inclusions are when the two estimates coincide, if any.

A customary trick to attack these problems is the assumption of parallel torsion. To be more precise, throughout this paper we shall be interested in connected Riemannian spin manifolds (M^n, g) endowed with a non-integrable G -structure ($G \subsetneq \mathrm{SO}_n$) and its *characteristic connection* ∇^c . This, if it exists, is a G -invariant metric connection with non-trivial skew-torsion T (unique in general), preserving the tensor fields defining the G -structure (we refer to [2] for a detailed exposition). Since for $s = 1/4$ the family ∇^s has torsion T , from now on we set $\nabla^{1/4} = \nabla^c$. Notice that our requirement $\nabla^c T = 0$ is not a very restrictive condition (see [29, 2] for several examples of G -structures satisfying this setting). On the other hand, it implies the vanishing of the co-differential $\delta T = 0$ and ensures the compatibility of the actions of ∇^c and T on Σ . Hence, the Ricci tensors Ric^c and more general Ric^s , remain symmetric and the same time the spinor bundle Σ decomposes into a direct sum of eigenbundles Σ_γ corresponding to the T -action.

In this note, we first compute the Ricci tensor and scalar curvature of a manifold (M^n, g, T) carrying twistor spinors with respect to the family ∇^s , under the assumption $\nabla^c T = 0$. Then, we prove that the twistor equation $P^s(\varphi) = 0$ can be viewed as a parallelism condition with respect to a suitable connection $\nabla^{s,E}$ on the vector bundle $\Sigma \oplus \Sigma$ (Theorem 2.5). In particular, a section $\varphi \in \Gamma(\Sigma)$ is a twistor spinor with torsion if and only if the section $\varphi \oplus D^s(\varphi)$ of $\Sigma \oplus \Sigma$ is $\nabla^{s,E}$ -parallel. In this way we generalize the Riemannian case (compare with [15, Thm. 4, p. 25], for instance) and we show that the zeros of such spinors are isolated.

Next, we take advantage of the splitting $\Sigma = \bigoplus_\gamma \Sigma_\gamma$ and examine a special class of TsT, namely these which are parallel under the characteristic connection ∇^c . Notice that the existence of a ∇^c -parallel spinor is already a strong condition which imposes restrictions on the holonomy group of ∇^c , see for example [24, 31, 29]. Here, for a compact triple (M^n, g, T) of positive scalar curvature, we present first a criterion which allows us to decide when a ∇^c -parallel spinor lying inside Σ_γ for some $\gamma \neq 0$, is a real Killing spinor (Proposition 3.2). For $3 < n \leq 8$ we see that this is the limiting case of the corresponding criterion related to its existence [3, Lem. 4.1]. Furthermore, under certain assumptions we identify these classes of spinors (Theorem 3.6) and next extend this correspondence to KsT and TsT as well. In particular, for a ∇^c -parallel spinor φ we show that the Riemannian Killing spinor equation $\nabla_X^g \varphi = \kappa X \cdot \varphi$ with $\kappa := 3\gamma/4n$ for some $\gamma \neq 0$ is equivalent to the KsT equation (*) for some (and thus any) $s \neq 0, \frac{1}{4}$ with Killing number $\zeta := 3(1-4s)\gamma/4n$ and moreover with the twistor equation (**) for some (and thus any) $s \neq 1/4$, under the additional condition $\varphi \in \Sigma_\gamma$ (Theorem 3.7). Using these results and combining with [31, Thm. 3.4] we finally conclude that on a compact Riemannian manifold of constant scalar curvature given by $\mathrm{Scal}^g = \frac{9(n-1)\gamma^2}{4n}$ for some non-zero T -eigenvalue $0 \neq \gamma \in \mathrm{Spec}(T)$, the following classes of spinors, if existent, coincide

$$\mathrm{Ker}(\nabla^c) \cong \bigoplus_\gamma \left[\Gamma(\Sigma_\gamma) \cap \mathcal{K}(M, g)_{\frac{3\gamma}{4n}} \right] \cong \bigoplus_\gamma \left[\Gamma(\Sigma_\gamma) \cap \mathcal{K}^s(M, g)_{\frac{3(1-4s)\gamma}{4n}} \right] \cong \bigoplus_\gamma \left[\mathrm{Ker}(P^s|_{\Sigma_\gamma}) \cap \mathrm{Ker}(D^c) \right],$$

under the assumption that $\nabla^c T = 0$ and that the symmetric endomorphism $dT + \frac{1}{2} \left[\frac{9(n-1)}{4n} \gamma^2 - \frac{3}{2} \|T\|^2 \right]$ acts on Σ with non-negative eigenvalues (see Theorem 3.6, Corollary 3.10).

In the following we examine the geometric constraints that imposes the existence of such spinors. Due to the previous correspondence, it is obvious that a triple (M^n, g, T) ($n \geq 3$) carrying a non-trivial ∇^c -parallel spinor field $\varphi \in \Gamma(\Sigma)$ which is a Killing spinor with torsion (KsT) with respect to the family $\nabla^s = \nabla^g + 2sT$ with Killing number $\zeta = 3\gamma(1-4s)/4n \neq 0$, must be Einstein

$$\mathrm{Ric}^g = \frac{9(n-1)\gamma^2}{4n^2} \mathrm{Id}_{TM}. \quad (\text{I})$$

Moreover, the equation $T \cdot \varphi = \gamma \varphi$ needs to hold. However, one can say much more; we prove that (M^n, g, T) is also ∇^c -Einstein (in fact, for $n = 3$ such an manifold must be Ric^c -flat, although never Ric^g -flat, see Proposition 5.1)

$$\mathrm{Ric}^c = \frac{3(n-3)\gamma^2}{n^2} \mathrm{Id}_{TM}. \quad (\text{II})$$

To do this, we follow a spinorial approach to ∇^c -Einstein manifolds carrying a ∇^c -parallel spinor $\varphi \in \Sigma_\gamma$, which is available for any triple (M^n, g, T) admitting parallel spinors $\varphi \in \Sigma_\gamma$ with respect to a metric connection ∇ with *parallel* skew-torsion T (see for example [29]). In our case, this technique allows us to deduce that (M^n, g, T) is ∇^c -Einstein and then we use this result to provide an alternative proof

of the original Einstein condition, independent of the fact that φ must be a real Killing spinor. In a sense, this is the opposite of the way that ∇^c -Einstein structures have been traditionally examined, especially in dimensions 6 and 7 (see e.g. [31, Prop. 10.4]), but also in more general cases, e.g. naturally reductive spaces (see [22]). Taking advantage of our study on twistor spinors we finally deduce that a triple (M^n, g, T) endowed with a ∇^c -parallel KsT with $\zeta = 3\gamma(1 - 4s)/4n \neq 0$, must be ∇^s -Einstein (with non-parallel torsion) for any $s \neq 0, 1/4$. In particular, the equation $\text{Ric}^s = \frac{\text{Scal}^s}{n}g$ is satisfied for any s , with the values $s = 0, 1/4$ being the special values described above (Proposition 5.1). Hence, *the existence of ∇^c -parallel real Killing spinor with Killing number $\kappa = 3\gamma/4n$ for some $0 \neq \gamma \in \text{Spec}(T)$, or equivalent the existence of a ∇^c -parallel KsT for some $s \neq 0, 1/4$ with $\zeta = 3\gamma(1 - 4s)/4n$, implies that*

$$\text{Ric}^s = \frac{\text{Scal}^s}{n}g, \quad \forall s \in \mathbb{R}. \quad (\text{III})$$

In this point, one should emphasise that our results have been described with respect to the same Riemannian metric (without any deformation). In particular, Theorem 3.6, Theorem 3.7 and Proposition 5.1 highlight this case and allow us to provide new examples (related with the existence of KsT). There are two representative classes of non-integrable G -structures carrying this special kind of KsT; nearly parallel G_2 -manifolds in dimension 7 and nearly Kähler manifolds in dimension 6. Another remarkable example is the round 3-sphere S^3 , where ∇^c is not unique and coincides with the flat ± 1 -connections of Cartan-Schouten. However, we point out that Einstein-Sasakian manifolds in any odd dimension ≥ 5 cannot be candidates of Theorem 3.7 for example, since according to [5, Rem. 2.26] such a manifold is never ∇^c -Einstein. Indeed, the integrability conditions (I) and (II) are already very strong and provide a recipe to describe several special structures endowed with their characteristic connection that fail to carry ∇^c -parallel Killing spinors with torsion with respect to $\nabla^s = \nabla^g + 2sT$. To avoid confusions, we mention that any Einstein-Sasaki manifold M^n ($n \geq 5$) carries KsT and these occur of the real Killing spinors after applying the Tanno deformation on the Einstein-Sasaki metric, see [10, Ex. 3.16] or [3, Ex. 5.1, 5.2] and for the full picture the Phd thesis [17]. However, for these spinors the parallelism equation and the KsT equation do not hold anymore with respect to the same metric. This is the main difference with the KsT lying at the heart of the present work; they are always parallel under the characteristic connection and both our spinorial equations are verified with respect to the same metric for which the equation $\nabla^c\varphi = 0$ holds.

For nearly parallel G_2 -structures, Theorem 3.7 gives rise to a complete description of all non-trivial ∇^c -parallel KsT that such a manifold admits (Theorem 4.2). The same applies in dimension 6 for nearly Kähler manifolds, with the difference that the result was known for $s = 5/12$, see [3, Thm. 6.1] or [17]. Here, we extend this correspondence to any parameter $s \in \mathbb{R} \setminus \{0, 1/4\}$ (Theorem 4.1). Notice that the existence of such spinors on nearly parallel G_2 -manifolds has been conjectured in [3], and this was part of our motivation. We finally remark that these weak holonomy structures are both solutions of the equations for the common sector of type II superstring theory: $\nabla^c\varphi = 0$, $T \cdot \varphi = \gamma \cdot \varphi$, $\delta(T) = 0$ and $\nabla^c \text{Ric}^c = 0$ (see [31, 1, 11, 2]). The supersymmetries of the model are interpreted by the ∇^c -parallel spinors. Our results show that there are models of this kind, which give rise to solutions of the Killing spinor equation with torsion and satisfy equation (III), for any $s \in \mathbb{R}$.

In the last part of the paper, we examine some further applications. We show for example that the integrability condition related to the existence of general KsT (for $s = (n-1)/4(n-3)$, see [3, Thm. A.2]), reduces in the special case of a ∇^c -parallel KsT with $\zeta = 3\gamma(1 - 4s)/4n$ to an identity, namely the twistor equation with torsion (Corollary 6.3). After that, we focus on the inequality $\beta_{\text{tw}}(\gamma) \leq \beta_{\text{univ}}(\gamma)$. For $3 < n \leq 8$ and for a spinor $\varphi \in \Sigma_\gamma$ satisfying the equation $\nabla^c\varphi = 0$, we prove that the equation $\beta_{\text{tw}}(\gamma) = \beta_{\text{univ}}(\gamma)$ is equivalent to say that φ is a real Killing spinor with Killing number $\kappa = 3\gamma/4n$ (Proposition 6.4). As a consequence, in dimension six this special case is exhausted by nearly Kähler manifolds (see [3, Ex. 6.1]) and in dimension 7 by nearly parallel G_2 -manifolds.

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1. PRELIMINARIES

Let us consider a connected Riemannian spin manifold (M^n, g, T) ($n = \dim M \geq 3$), carrying a non-trivial 3-form $T \in \bigwedge^3 M$, as in introduction.¹ Recall that for some $s \in \mathbb{R}$, ∇^s is the metric connection with skew-torsion $4sT$. This is defined by

$$g(\nabla_X^s Y, Z) = g(\nabla_X^g Y, Z) + 2sT(X, Y, Z)$$

and joins the characteristic connection $\nabla^{1/4} \equiv \nabla^c$ with the Levi-Civita connection $\nabla^0 \equiv \nabla^g$. A Riemannian manifold endowed with a metric connection $\nabla \equiv \nabla^c$ with totally skew-symmetric torsion T , is usually called Riemannian manifold with torsion and the geometry Riemann-Cartan geometry (see for example [24]). Here we will not use this notation and whenever we refer to a triple (M^n, g, T) we shall mean a connected Riemannian spin manifold endowed with the above family of connections. It is useful to consider the 4-form

$$\sigma_T := \frac{1}{2} \sum_{i=1}^n (e_i \lrcorner T) \wedge (e_i \lrcorner T)$$

and normalize the length of the 3-form T as $\|T\|^2 = \frac{1}{3} \sum_{i \leq i < j \leq n} g(T(e_i, e_j), T(e_i, e_j))$, where $\{e_1, \dots, e_n\}$ is an orthonormal frame of (M^n, g) . The Riemannian scalar curvature Scal^g and the scalar curvature induced by ∇^s are connected by the rule $\text{Scal}^s = \text{Scal}^g - 24s^2\|T\|^2$. Moreover, inside the Clifford algebra we have that $2\sigma_T = \|T\|^2 - T^2$, for details and proofs see [24, 40, 31, 1, 6, 2, 3]. The lift of ∇^s to the spinor bundle Σ is given by

$$\nabla_X^s \varphi = \nabla_X^g \varphi + s(X \lrcorner T) \cdot \varphi$$

for any $X \in \Gamma(TM)$ and $\varphi \in \Gamma(\Sigma)$. After identifying $TM \cong T^*M$ via the metric tensor, the associated Dirac operator $D^s : \Gamma(\Sigma) \xrightarrow{\nabla^s} \Gamma(TM \otimes \Sigma) \xrightarrow{\mu} \Gamma(\Sigma)$ reads

$$D^s(\varphi) = \sum_i e_i \cdot \nabla_{e_i}^s \varphi = \sum_i e_i \cdot (\nabla_{e_i}^g \varphi + s(e_i \lrcorner T) \cdot \varphi) = D^g(\varphi) + 3sT \cdot \varphi,$$

where $D^g \equiv D^0 := \mu \circ \nabla^0$ is the Riemannian Dirac operator. For some $s \in \mathbb{R} \setminus \{0\}$ we shall denote by

$$\mathcal{K}^s(M, g)_\zeta := \{\varphi \in \Gamma(\Sigma) : \nabla_X^s \varphi = \zeta X \cdot \varphi \quad \forall X \in \Gamma(TM)\}$$

the set of all *Killing spinors with torsion* (KsT) with respect to the family $\nabla^s = \nabla^g + 2sT$ with Killing number $\zeta \neq 0$. Similarly, $\mathcal{K}(M, g)_\kappa$ will denote the set of Riemannian Killing spinors with Killing number $\kappa \neq 0$. In general, the Killing number ζ can be a complex number; however only solutions with $\zeta \in \mathbb{R} \setminus \{0\}$ are known (see also [10]). Next, we are mainly interested in real Killing numbers (with torsion or not). Notice also that we do *not* view ∇^s -parallel spinor as a special case of a KsT. We finally remark that in [3, Def. 5.1], the parameter s is “fixed”, in the sense that it depends on $n = \dim M$, namely $s = (n-1)/4(n-3)$, see [3] for more explanations. Here we relax this condition and follow the definition of [10], which is more general. Let us also recall that

Definition 1.1. ([3]) A *twistor spinor with torsion* (TsT) is a spinor field solving the differential equation $\nabla_X^s \varphi + (1/n)X \cdot D^s(\varphi) = 0$, for any $X \in \Gamma(TM)$, i.e. an element in the kernel of the Penrose or twistor operator P^s associated to ∇^s . This is the differential operator defined by the composition

$$\Gamma(\Sigma) \xrightarrow{\nabla^s} \Gamma(TM \otimes \Sigma) \xrightarrow{p} \Gamma(\ker \mu), \quad P^s = p \circ \nabla^s,$$

where $p : TM \otimes \Sigma \rightarrow \ker \mu \subset TM \otimes \Sigma$ is the orthogonal projection onto the kernel of the Clifford multiplication. Locally one has

$$p(X \otimes \varphi) := X \otimes \varphi + \frac{1}{n} \sum_{i=1}^n e_i \otimes e_i \cdot X \cdot \varphi, \quad P^s \varphi := \sum_{i=1}^n e_i \otimes \{\nabla_{e_i}^s \varphi + \frac{1}{n} e_i \cdot D^s \varphi\}.$$

Any KsT with Killing number $\zeta \neq 0$ is a D^s -eigenspinor, i.e. $D^s \varphi = -n\zeta \varphi$ and thus a special solution of the twistor equation with torsion. And conversely, any twistor spinor $\varphi \in \text{Ker}(P^s)$ which is the same time a D^s -eigenspinor, is also a KsT. It is easy to see that TsT are satisfying the same basic properties with Riemannian twistor spinors [3, Th. 3.1]. In the following we shall develop a theory for TsT, as an analogue of the Riemannian case.

¹In this paper all the manifolds, tensor fields and other geometric objects under consideration, are assumed to be smooth.

2. TWISTOR SPINORS WITH TORSION

2.1. Twistor spinors with torsion. From now on, and for the following of this article we assume that the characteristic connection $\nabla^c := \nabla^g + (1/2)T$ satisfies the equation $\nabla^c T = 0$. Then, the relations $\delta T = 0$ and $dT = 2\sigma_T$ hold and the length $\|T\|^2$ is constant, see [6, 2]. Moreover, the curvature tensor R^s is symmetric $R^s(X, Y, Z, W) = R^s(Z, W, X, Y)$ and the same holds for the Ricci tensor $\text{Ric}^s(X, Y) = \sum_i R^s(X, e_i, e_i, Y)$ (since $\delta^s T = 0$, see [3, Thm. B.1]). In fact, one can write $\text{Ric}^s(X, Y) = \text{Ric}^g(X, Y) - 4s^2 S(X, Y)$, where S is the symmetric tensor given by $S(X, Y) := \sum_i g(T(X, e_i), T(Y, e_i))$. This easily occurs by combining for example the formulas [5, p. 740]

$$\begin{aligned} R^s(X, Y, Z, W) &= R^g(X, Y, Z, W) + 4s^2 [g(T(X, Y), T(Z, W)) + \sigma_T(X, Y, Z, W)] \\ &\quad + 2s [\nabla_X^s T(Y, Z, W) - \nabla_Y^s T(X, Z, W)]. \end{aligned}$$

and $\nabla_X^s T(Y, Z, V) = \frac{4s-1}{2} \sigma_T(Y, Z, V, X)$ [3, Thm. B.1]. Passing now to the spinor bundle Σ , the curvature tensor R^s associated to the lift of ∇^s is defined by $R^s(X, Y)\varphi = \nabla_X^s \nabla_Y^s \varphi - \nabla_Y^s \nabla_X^s \varphi - \nabla_{[X, Y]}^s \varphi$. In her Phd thesis, J. Becker-Bender proved that the Ricci endomorphism $\text{Ric}^s(X) := \sum_i \text{Ric}^s(X, e_i) e_i \cdot \varphi$ is related with R^s and σ_T as follows (see also [3, Thm. A.1] for $s = 1/4$):

Lemma 2.1. ([17, Lem. 1.13])

$$\sum_i e_i \cdot R^s(X, e_i) \varphi = -\frac{1}{2} \text{Ric}^s(X) \cdot \varphi + s(3 - 4s)(X \lrcorner \sigma_T) \cdot \varphi, \quad \forall \varphi \in \Gamma(\Sigma), X \in \Gamma(TM).$$

We present now the Ricci curvature associated to the family ∇^s , on a manifold (M^n, g, T) carrying a twistor spinor with torsion w.r.t. ∇^s , i.e. $\nabla_X^s \varphi = -\frac{1}{n} X \cdot D^s(\varphi)$, under the condition $\nabla^c T = 0$.

Lemma 2.2. *For any twistor spinor $\varphi \in \text{Ker}(P^s)$ and for any vector field X the following relations hold:*

$$\begin{aligned} -\frac{1}{2} \text{Ric}^s(X) \cdot \varphi &= -\frac{8s}{n} (X \lrcorner T) \cdot D^s(\varphi) + \frac{n-2}{n} \nabla_X^s (D^s(\varphi)) - \frac{1}{n} X \cdot (D^s)^2(\varphi) - s(3 - 4s)(X \lrcorner \sigma_T) \cdot \varphi. \\ \frac{1}{2} \text{Scal}^s \varphi &= -\frac{24s}{n} T \cdot D^s(\varphi) + \frac{2(n-1)}{n} (D^s)^2(\varphi) - 4s(3 - 4s) \sigma_T \cdot \varphi. \end{aligned}$$

Proof. Consider a twistor spinor $\varphi \in \text{Ker}(P^s)$ for some $s \in \mathbb{R}$. For some $X, Y \in \Gamma(TM)$ it is

$$\begin{aligned} \nabla_X^s \nabla_Y^s \varphi &= -\frac{1}{n} (\nabla_X^s Y) \cdot D^s(\varphi) - \frac{1}{n} Y \cdot \nabla_X^s (D^s(\varphi)), \\ \nabla_Y^s \nabla_X^s \varphi &= -\frac{1}{n} (\nabla_Y^s X) \cdot D^s(\varphi) - \frac{1}{n} X \cdot \nabla_Y^s (D^s(\varphi)). \end{aligned}$$

Hence, for the curvature tensor on Σ we compute

$$R^s(X, Y)\varphi = -\frac{1}{n} T^s(X, Y) \cdot D^s(\varphi) - \frac{1}{n} [Y \cdot \nabla_X^s (D^s(\varphi)) - X \cdot \nabla_Y^s (D^s(\varphi))],$$

where $T^s(X, Y) := \nabla_X^s Y - \nabla_Y^s X - [X, Y] = 4sT(X, Y)$ is the torsion of the family ∇^s . Let $\{e_1, \dots, e_n\}$ a local orthonormal frame of TM . Then, multiplying by e_i and summing we conclude that

$$\begin{aligned} \sum_i e_i \cdot R^s(X, e_i) \varphi &= -\frac{4s}{n} \sum_i e_i \cdot T(X, e_i) \cdot D^s(\varphi) - \frac{1}{n} \sum_i e_i \cdot e_i \cdot \nabla_X^s (D^s(\varphi)) \\ &\quad + \frac{1}{n} \sum_i e_i \cdot X \cdot \nabla_{e_i}^s (D^s(\varphi)) \\ &= -\frac{8s}{n} (X \lrcorner T) \cdot D^s(\varphi) + \nabla_X^s (D^s(\varphi)) + \frac{1}{n} \sum_i e_i \cdot X \cdot \nabla_{e_i}^s (D^s(\varphi)), \end{aligned}$$

where the relation $\sum_i e_i \cdot T(X, e_i) = 2(X \lrcorner T)$ was used (see [3, p. 325]). Recalling that $e_i \cdot X + X \cdot e_i = -2g(e_i, X) \text{Id}_\Sigma$, we also get $\frac{1}{n} \sum_i e_i \cdot X \cdot \nabla_{e_i}^s (D^s(\varphi)) = -\frac{1}{n} X \cdot (D^s)^2(\varphi) - \frac{2}{n} \nabla_X^s (D^s(\varphi))$. Therefore, for any $\varphi \in \text{Ker}(P^s)$ the following holds:

$$\sum_i e_i \cdot R^s(X, e_i) \varphi = -\frac{8s}{n} (X \lrcorner T) \cdot D^s(\varphi) + \frac{n-2}{n} \nabla_X^s (D^s(\varphi)) - \frac{1}{n} X \cdot (D^s)^2(\varphi).$$

Now, our first assertion is an immediate consequence of Lemma 2.1. We proceed with the scalar curvature. Since Ric^s is symmetric, we have that

$$\sum_i e_i \cdot \text{Ric}^s(e_i) \cdot \varphi = \sum_{i,j} \text{Ric}^s(e_i, e_j) \cdot e_i \cdot e_j \cdot \varphi = - \sum_i \text{Ric}^s(e_i, e_i) \cdot \varphi = - \text{Scal}^c \cdot \varphi.$$

But then, combining with the expression of Ric^s and observing that $\sum_i e_i \cdot (e_i \lrcorner T) = 3T$, $\sum_i e_i \cdot (e_i \lrcorner \sigma_T) = 4\sigma_T$ and $\sum_i e_i \cdot \nabla_{e_i}^s (D^s(\varphi)) = (D^s)^2(\varphi)$, we easily finish the proof. Notice that the given formula of Scal^s , encodes also the action of the square of the Dirac operator D^s on twistor spinors. Indeed, in our case $\nabla^c T = 0$ it is well-known that the following holds (see [6, Thm. 6.1] or [3, Thm. 2.1])

$$(D^s)^2(\varphi) = \Delta^s(\varphi) + s(3 - 4s)dT \cdot \varphi - 4s\mathcal{D}^s(\varphi) + \frac{1}{4}\text{Scal}^c \varphi,$$

where $\mathcal{D}^s := \sum_i (e_i \lrcorner T) \cdot \nabla_{e_i}^s \varphi$. But the action of the differential operators $\Delta^s := (\nabla^s)^* \nabla^s$ and \mathcal{D}^s on some $\varphi \in \text{Ker}(P^s)$, is given by

$$\mathcal{D}^s(\varphi) = \sum_i (e_i \lrcorner T) \cdot \nabla_{e_i}^s \varphi = -\frac{1}{n} \sum_i (e_i \lrcorner T) \cdot e_i \cdot D^s(\varphi) = -\frac{3}{n} T \cdot D^s(\varphi)$$

and $\Delta^s(\varphi) = \frac{1}{n}(D^s)^2(\varphi)$, respectively. Hence, for a twistor spinor with torsion we obtain that

$$\begin{aligned} (D^s)^2(\varphi) &= \frac{n}{n-1} \left[s(3-4s)dT \cdot \varphi - 4s\mathcal{D}^s(\varphi) + \frac{1}{4}\text{Scal}^s \varphi \right] \\ &= \frac{n}{n-1} \left[s(3-4s)dT \cdot \varphi + \frac{12s}{n} T \cdot D^s(\varphi) + \frac{1}{4}\text{Scal}^s \varphi \right], \quad (\mathfrak{h}) \end{aligned}$$

which is equivalent with the given expression of Scal^s (observing that $dT = 2\sigma_T$). \square

Remarks: For $s = 0$, i.e. for the Riemannian connection (zero torsion $T \equiv 0$), Lemma 2.2 reduces to a basic result of A. Lichnerowicz [42] about Riemannian twistor spinors (see also [15, pp. 23-24], [28, pp. 122-123] or [36, p. 134]):

$$\nabla_X^g(D^g(\varphi)) = \frac{n}{2(n-2)} \left[-\text{Ric}^g(X) \cdot \varphi + \frac{\text{Scal}^g}{2(n-1)} X \cdot \varphi \right] = \frac{n}{2} \text{Sch}^g(X) \cdot \varphi, \quad (D^g)^2(\varphi) = \frac{n \text{Scal}^g}{4(n-1)} \varphi,$$

where $\text{Sch}^g(X) := \frac{1}{n-2} [-\text{Ric}^g(X) \cdot \varphi + \frac{\text{Scal}^g}{2(n-1)} X]$ is the endomorphism induced by the Schouten tensor of ∇^g . For the family ∇^s , using (h) and Lemma 2.2, we conclude that any element in $\text{Ker}(P^s)$ satisfies the more general formula

$$\begin{aligned} \nabla_X^s(D^s(\varphi)) &= \frac{n}{2} \text{Sch}^s(X) \cdot \varphi + \frac{sn}{(n-1)(n-2)} \left[\left(\frac{8(n-1)}{n} (X \lrcorner T) + \frac{12}{n} X \cdot T \right) \cdot D^s(\varphi) \right. \\ (2.1) \quad &\quad \left. + (3-4s)(X \cdot dT + (n-1)(X \lrcorner \sigma_T)) \cdot \varphi \right], \end{aligned}$$

where here, Sch^s is the Schouten endomorphism associated to ∇^s . Notice that the formula for $(D^g)^2$ immediately appears by (h) for $s = 0$, which is its analogue for connections with parallel skew-torsion.

In the case that $\varphi \in \Gamma(\Sigma)$ is a Killing spinor with torsion, Lemma 2.2 applies and gives rise to the Ricci tensor Ric^s and the scalar curvature Scal^s of a Riemannian manifold carrying a Killing spinor with torsion with respect to ∇^s (as in the Riemannian case, too). Hence, through a method based on twistor spinors, one can now verify formulas that are already known by [17, Lem. 1.14]. We cite them in a corollary below.

Corollary 2.3. ([17, Lem. 1.14]) *Given a Killing spinor with torsion $\varphi \in \Gamma(\Sigma)$ w.r.t. the family $\nabla^s = \nabla^g + 2sT$ and with Killing number $\zeta \in \mathbb{R} \setminus \{0\}$, the following hold:*

$$\begin{aligned} \text{Ric}^s(X) \cdot \varphi &= 4(n-1)\zeta^2 X \cdot \varphi - 16s\zeta(X \lrcorner T) \cdot \varphi + 2s(3-4s)(X \lrcorner \sigma_T) \cdot \varphi, \\ \text{Scal}^s \varphi &= 4n(n-1)\zeta^2 \varphi + 48s\zeta T \cdot \varphi - 8s(3-4s)\sigma_T \cdot \varphi. \end{aligned}$$

Proof. It is an immediate consequence of Lemma 2.2. For a direct proof we refer to [17]. \square

Now, in a similar way with the Riemannian case, equation (2.1) tell us that the twistor equation with torsion can be viewed as a parallelism equation with respect to a suitable covariant derivative on the bundle $E := \Sigma \oplus \Sigma$. But let us explain this generalisation in full details.

Lemma 2.4. Consider the mapping $\nabla^{s,E} : \Gamma(E) \rightarrow \Gamma(T^*M \otimes E)$ given by

$$\begin{aligned} \nabla_X^{s,E}(\varphi_1 \oplus \varphi_2) &= \left(\nabla_X^s \varphi_1 + \frac{1}{n} X \cdot \varphi_2 \right) \oplus \left(-\frac{n}{2} \text{Sch}^s(X) \cdot \varphi_1 - \frac{sn(3-4s)}{(n-1)(n-2)} [X \cdot dT + (n-1)(X \lrcorner \sigma_T)] \cdot \varphi_1 \right. \\ &\quad \left. - \frac{sn}{(n-1)(n-2)} \left[\frac{8(n-1)}{n} (X \lrcorner T) + \frac{12}{n} X \cdot T \right] \cdot \varphi_2 + \nabla_X^s \varphi_2 \right), \end{aligned}$$

for any $X \in \Gamma(TM)$ and $\varphi_1 \oplus \varphi_2 \in \Gamma(\Sigma \oplus \Sigma)$. Then, $\nabla^{s,E}$ defines a covariant derivative on the vector bundle $E = \Sigma \oplus \Sigma$.

Proof. The linearity of $\nabla^{s,E}$ is obvious. We need only to check the rule

$$\nabla^{s,E}(f(\varphi_1 \oplus \varphi_2)) = df \otimes (\varphi_1 \oplus \varphi_2) + f \nabla^{s,E}(\varphi_1 \oplus \varphi_2),$$

for some smooth function $f \in C^\infty(M; \mathbb{R})$. But this is a simple consequence of relation $\nabla_X^s(f\varphi_i) = X(f)\varphi_i + f\nabla_X^s\varphi_i = df(X) \otimes \varphi_i + f\nabla_X^s\varphi_i$ and the fact that all the other parts are tensors (differential forms) acting by Clifford multiplication. \square

Notice that for $s = 0$, the connection $\nabla^{0,E} \equiv \nabla^E$ coincides with the connection ∇^E described in [27, p. 61] (see also [15, p. 25]). We conclude that

Theorem 2.5. Let (M^n, g, T) ($n \geq 3$) a connected Riemannian spin manifold with $\nabla^c T = 0$. Then, any twistor spinor with torsion $\varphi \in \text{Ker}(P^s)$ satisfies the equation $\nabla_X^{s,E}(\varphi \oplus D^s(\varphi)) = 0$. Conversely, if $(\varphi \oplus \psi) \in \Gamma(E)$ is $\nabla^{s,E}$ -parallel, then φ is a twistor spinor with torsion such that $D^s(\varphi) = \psi$.

Proof. By definition,

$$\nabla_X^{s,E} = \begin{pmatrix} \nabla_X^s & (1/n)X \\ -\frac{n}{2} \text{Sch}^s(X) - \frac{sn(3-4s)}{(n-1)(n-2)} [X \cdot dT + (n-1)(X \lrcorner \sigma_T)] & -\frac{sn}{(n-1)(n-2)} \left[\frac{8(n-1)}{n} (X \lrcorner T) + \frac{12}{n} X \cdot T \right] + \nabla_X^s \end{pmatrix}.$$

The twistor equation in combination with (2.1) implies now the result: $\nabla_X^{s,E} \begin{pmatrix} \varphi \\ D^s(\varphi) \end{pmatrix} = 0$. The converse occurs due to the relation $\nabla_{e_i}^s \varphi + \frac{1}{n} e_i \cdot \psi = 0$, after multiplying with e_i and adding, see also [15, p. 25] or [36, p. 136] for $s = 0$. \square

Thus, a section $\varphi \in \Gamma(\Sigma)$ is a twistor spinor with torsion if and only if the section $\varphi \oplus D^s(\varphi)$ of $\Sigma \oplus \Sigma$ is $\nabla^{s,E}$ -parallel. Consequently, any element $\varphi \in \text{Ker}(P^s)$ is defined by its values $\varphi_p, (D^s(\varphi))_p$ at some point $p \in M$. Hence

Corollary 2.6. Let (M^n, g, T) ($n \geq 3$) a connected Riemannian spin manifold with $\nabla^c T = 0$. Then,

- a) The kernel of the twistor operator is a finite dimensional space, i.e. $\dim_{\mathbb{C}} \text{Ker}(P^s) \leq 2^{\lfloor \frac{n}{2} \rfloor + 1}$.
- b) If φ and $D^s(\varphi)$ vanish at some point $p \in M$ and $\varphi \in \text{Ker}(P^s)$, then $\varphi \equiv 0$.

Proof. Both are consequences of Theorem 2.5. Notice that $\text{rk}_{\mathbb{C}}(\Sigma) = 2^{\lfloor \frac{n}{2} \rfloor}$ and that parallel sections on vector bundles over connected manifolds are uniquely determined by their value in a single point. \square

Using Theorem 2.5 we present the corresponding version of a classical property related with the zeros of a twistor spinor, see [27, 15].

Remarks: Locally, one may assume that $(\nabla^s X)_p = 0$ at some point $p \in M$ (or in other words $(\nabla^s e_j)_p = 0$ for some local orthonormal frame of TM). We mention that this condition doesn't mean that locally the torsion $T^s = 4sT$ vanishes, see for example [31, p. 307].

Proposition 2.7. Let (M^n, g, T) ($n \geq 3$) be a connected Riemannian spin manifold with $\nabla^c T = 0$. Then any zero point of a twistor spinor with torsion $0 \neq \varphi \in \text{Ker}(P^s)$ is isolated, i.e. the zero-set of φ is discrete.

Proof. Fix some $p \in M$. We compute the Hessian Hess^{∇^s} of the function $|\varphi|^2$ in p . We shall denote by $(\cdot, \cdot) := \text{Re} \langle \cdot, \cdot \rangle$ the real inner product on Σ induced by the corresponding Hermitian inner product $\langle \cdot, \cdot \rangle$

on Σ . For a twistor spinor $\varphi \in \text{Ker}(P^s)$ we compute $X(|\varphi|^2) = 2(\nabla_X^s \varphi, \varphi) = -\frac{2}{n}(X \cdot D^s(\varphi), \varphi)$. Thus, in combination with the previous remark we conclude that locally the following holds:

$$\begin{aligned} \text{Hess}_p^{\nabla^s}(|\varphi|^2)(X, Y) &= (XY(|\varphi|^2))_p = -\frac{2}{n} \left[X(Y \cdot D^s(\varphi), \varphi) \right]_p \\ &= -\frac{2}{n} \left[(\nabla_X^s(Y \cdot D^s(\varphi)), \varphi) + (Y \cdot D^s(\varphi), \nabla_X^s \varphi) \right]_p \\ &= -\frac{2}{n} \left[(Y \cdot \nabla_X^s(D^s(\varphi)), \varphi) \right]_p + \frac{2}{n^2} \left[(Y \cdot D^s(\varphi), X \cdot D^s(\varphi)) \right]_p. \end{aligned}$$

Based on (2.1) we show that the first term $-\frac{2}{n} \left[(Y \cdot \nabla_X^s(D^s(\varphi)), \varphi) \right]_p$ vanishes, under the assumption $\varphi_p = 0$. The cancelation of the part related with the Schouten endomorphism is explained as follows:

$$\begin{aligned} -\left(\frac{1}{(n-2)} Y \cdot \left[-\text{Ric}^s(X) + \frac{\text{Scal}^s}{2(n-1)} X \right] \cdot \varphi, \varphi \right) &= \frac{1}{n-2} (Y \cdot \text{Ric}^s(X) \cdot \varphi, \varphi) - \frac{\text{Scal}^s}{(n-1)(n-2)} (Y \cdot X \cdot \varphi, \varphi) \\ &= -\frac{|\varphi|^2}{n-2} \text{Ric}^s(X, Y) + \frac{|\varphi|^2 \text{Scal}^s}{(n-1)(n-2)} g(X, Y). \end{aligned}$$

Here, we use $\text{Ric}^s(X) := \sum_i \text{Ric}^s(X, e_i) e_i \cdot \varphi$ and the general formula $(X \cdot \varphi, Y \cdot \varphi) = g(X, Y) |\varphi|^2$. Similarly, for the next term we get

$$\frac{2s}{(n-1)(n-2)} \left(\left[\frac{8(n-1)}{n} (X \lrcorner T) + \frac{12}{n} X \cdot T \right] \cdot D^s(\varphi), Y \cdot \varphi \right),$$

which its evaluation in p equals to zero (since $Y \cdot \varphi_p = 0$ and $(\cdot, \cdot) = \text{Re} \langle \cdot, \cdot \rangle$) and the same is true for the last component. Hence finally, if $\varphi_p = 0$ we conclude that the Hessian is given by

$$\text{Hess}_p^{\nabla^s}(|\varphi|^2)(X, Y) = \frac{2}{n^2} \left[(Y \cdot D^s(\varphi), X \cdot D^s(\varphi)) \right]_p = \frac{2}{n^2} (|D^s(\varphi)|^2)_p g_p(X, Y).$$

The claim now follows as in the classical case [27, Prop. 2]; if $(D^s(\varphi))_p \neq 0$, then p is a non-degenerate critical point of $|\varphi|^2$ and thus an isolated zero point of φ . If $(D^s(\varphi))_p = 0$, then φ must be trivial by Corollary 2.6. \square

We conclude that twistor spinors with torsion satisfy the same structural properties with the original twistor spinors. This should not be a surprising result, since there are already known examples of twistor spinors associated to linear connections different than ∇^g which still satisfy these characteristic properties. For instance, for twistor spinors on Weyl manifolds or on manifolds with a foliated structure, i.e. transversal twistor spinors, results like Theorem 2.5, Corollary 2.6 and Proposition 2.7 are available, see [20, Thm. 2.7, 2.9], and [39, pp. 14-15], respectively. However, one has to stress that the (global) behaviour of the twistor or Killing spinor equation with torsion is in general different than the behaviour of their Riemannian analogues, in the sense that the existence of such spinors imposes different consequences on the manifold, depending on the geometry and the given G -structure (see [3, 17, 10]). Next we shall describe a very special class of twistor or Killing spinors with torsion, namely these which are ∇^c -parallel and the same time elements of some T -eigenspace Σ_γ with $\gamma \neq 0$.

3. SPINOR FIELDS PARALLEL UNDER THE CHARACTERISTIC CONNECTION

3.1. ∇^c -parallel spinors. Consider the cubic Dirac operator $\not{D} = D^g + \frac{1}{4}T$. For the square of \not{D} the following formula of Schrödinger-Lichnerowicz type is well-known (apply [3, Thm. 2.2] for $s = 1/4$ and see also [24, 6, 7])

$$\not{D}^2 = \Delta_T + \frac{1}{4} \text{Scal}^g - \frac{1}{4} T^2 + \frac{1}{8} \|T\|^2,$$

while for the square of D^c it holds that (see [31, Thm. 3.1])

$$(D^c)^2(\varphi) = \Delta_T(\varphi) + \frac{1}{2} dT \cdot \varphi - \sum_i (e_i \lrcorner T) \cdot \nabla_{e_i}^c \varphi + \frac{1}{4} \text{Scal}^c \cdot \varphi.$$

In both cases, $\Delta_T := (\nabla^c)^* \nabla^c := -\sum_i \nabla_{e_i}^c \nabla_{e_i}^c + \nabla_{\nabla_{e_i}^g e_i}^c$ denotes the spinor Laplacian associated to the connection ∇^c . Because $\nabla^c T = 0$, the square \not{D}^2 commutes with the symmetric endomorphism T . In particular, T acts on spinors with real constant eigenvalues [11, Thm. 1.1] and the spinor bundle decomposes into a direct sum of T -eigenbundles preserved by ∇^c : $\Sigma = \bigoplus_{\gamma \in \text{Spec}(T)} \Sigma_\gamma$ with $\nabla^c \Sigma_\gamma \subset \Sigma_\gamma$, $\forall \gamma \in \text{Spec}(T)$. Similarly, the space of sections decomposes as $\Gamma(\Sigma) = \bigoplus_{\gamma \in \text{Spec}(T)} \Gamma(\Sigma_\gamma)$. Then, from the generalized Schrödinger-Lichnerowicz formula and for the smallest eigenvalue λ_1 of the square \not{D}^2 restricted on Σ_γ it follows that (see [6, 3])

$$\lambda_1(\not{D}^2|_{\Sigma_\gamma}) \geq \frac{1}{4} \text{Scal}^g_{\min} + \frac{1}{8} \|T\|^2 - \frac{1}{4} \gamma^2 := \beta_{\text{univ}}(\gamma).$$

The equality occurs if and only if Scal^g is constant and φ is ∇^c -parallel, i.e. $\nabla_X^g \varphi + \frac{1}{4}(X \lrcorner T) \cdot \varphi = 0$. These are the spinors fields that we are mainly interested in here. A very similar formula with this estimate holds on the whole spinor bundle $\Sigma = \bigoplus_\gamma \Sigma_\gamma$, where one has to consider the maximum of the possible different eigenvalues $\{\gamma_1^2, \dots, \gamma_q^2\}$, see [3] for details.

∇^c -parallel or ∇^c -harmonic spinor fields have been originally studied in [31], see also [24, 40] for 4-dimensional manifolds or KT-manifolds (Kähler with torsion), [29] for G_2 -manifolds and [6, 7] for a description in terms of the Casimir operator. With the purpose of having the results in a uniform notation, we recall that

Theorem 3.1. ([31, Cor. 3.2], [11, Thm. 1.1], [3, p. 306]) *Let φ_0 be a ∇^c -parallel spinor. Then,*

$$\begin{aligned} \text{Scal}^c \varphi_0 &= -2dT \cdot \varphi_0 = -4\sigma_T \cdot \varphi_0, \quad (*) \\ \text{Ric}^c(X) \cdot \varphi_0 &= \frac{1}{2}(X \lrcorner dT + \nabla_X^c T) \cdot \varphi_0, \end{aligned}$$

in particular the scalar curvatures $\text{Scal}^c, \text{Scal}^g$ are constant. If in addition $\varphi \in \Sigma_\gamma$ for some $\gamma \neq 0$, then

$$\text{Scal}^g = 2\gamma^2 - \frac{1}{2} \|T\|^2, \quad \text{Scal}^c = 2(\gamma^2 - \|T\|^2), \quad \text{Ric}^c(X) = \frac{1}{2}(X \lrcorner dT) \cdot \varphi_0 = (X \lrcorner \sigma_T) \cdot \varphi_0.$$

For convenience, we shall often refer to $(*)$ as the *integrability condition* for the existence of a ∇^c -parallel spinor. We also agree to use the notation *characteristic spinor field*, for a spinor field in the kernel of the Dirac operator induced by ∇^c , i.e. a ∇^c -harmonic spinor field. Obviously, any ∇^c -parallel spinor is characteristic, but the converse is false in general [31, Thm. 3.4].

3.2. ∇^c -parallel spinors which are also Killing spinors. Let us provide now a necessary and sufficient condition which allows us to decide when a ∇^c -parallel spinor lying in Σ_γ is a real Killing spinor (with respect to the same metric g that the equation $\nabla_X^g \varphi + \frac{1}{4}(X \lrcorner T) \cdot \varphi = 0$ holds). For $n \leq 8$, we see that this is the limiting case of a criterion about the existence of this kind of spinors, see [3, Lem. 4.1] or the inequalities given in (6.4) in Section 6. In particular, we show that if such a spinor exists and one of these inequalities holds as an equality, then the spinor is a Killing spinor (see also Proposition 6.4).

Proposition 3.2. *Let (M^n, g, T) be a compact connected Riemannian spin manifold with $\nabla^c T = 0$ and positive scalar curvature, carrying a non-trivial spinor field $\varphi_0 \in \Gamma(\Sigma)$ such that*

$$(3.1) \quad \nabla_X^c \varphi_0 = \nabla_X^g \varphi_0 + \frac{1}{4}(X \lrcorner T) \cdot \varphi_0 = 0, \quad \forall X \in \Gamma(TM), \quad T \cdot \varphi_0 = \gamma \varphi_0, \quad \text{for some } \gamma \in \mathbb{R} \setminus \{0\}.$$

Then, φ_0 is a real Killing spinor (with respect to g) if and only if the (constant) eigenvalue $0 \neq \gamma \in \text{Spec}(T)$ satisfies the equation

$$\gamma^2 = \frac{4n}{9(n-1)} \text{Scal}^g. \quad (\dagger)$$

If this is the case, then the Killing number is given by $\kappa := 3\gamma/4n$ and the following relation hold

$$(3.2) \quad (X \lrcorner T) \cdot \varphi_0 + \frac{3\gamma}{n} X \cdot \varphi_0 = 0, \quad \forall X \in \Gamma(TM).$$

For $n \leq 8$, the condition (\dagger) is equivalent to $\gamma^2 = \frac{2n}{9-n} \|T\|^2$ or $\text{Scal}^g = \frac{9(n-1)}{2(9-n)} \|T\|^2$ and if this is the case, then the action of the symmetric endomorphism dT on φ_0 is given by $dT \cdot \varphi_0 = -\frac{3\gamma^2(n-3)}{2n} \varphi_0$.

Proof. Because $D^c \equiv D^{1/4} = D^g + \frac{3}{4}T$ and $\varphi_0 \in \Sigma_\gamma$ is ∇^c -parallel, it follows that φ_0 is an eigenspinor of the Riemannian Dirac operator D^g (and thus also of $\not{D} = D^g + \frac{1}{4}T$):

$$D^g(\varphi_0) = -\frac{3\gamma}{4}\varphi_0.$$

Given a compact Riemannian spin manifold with positive scalar curvature, it is well-known that if $\varphi_0 \in \Gamma(\Sigma)$ is an eigenspinor of D^g with one of the eigenvalues $\pm \frac{1}{2}\sqrt{\frac{n \text{Scal}^g}{n-1}}$, then φ_0 is a real Killing spinor with corresponding Killing number $\kappa := \mp \frac{1}{2}\sqrt{\frac{\text{Scal}^g}{n(n-1)}}$ (the converse is also true, see [15, pp. 20, 31] or [25, 38]). However, since φ_0 is ∇^c -parallel, Theorem 3.1 allows us to drop the minimal condition in Scal^g . Hence, let us assume for example that γ is positive; then φ_0 is a real Killing spinor if and only if

$$-\frac{3\gamma}{4} = -\frac{1}{2}\sqrt{\frac{n \text{Scal}^g}{n-1}},$$

which gives rise to the stated relation (†). Then, we also compute $\kappa = +\frac{1}{2}\sqrt{\frac{\text{Scal}^g}{n(n-1)}} = 3\gamma/4n$. Similar is treated the case where γ is negative, where one has $-\frac{3\gamma}{4} = \frac{1}{2}\sqrt{\frac{n \text{Scal}^g}{n-1}}$ and the Killing number must be given by $\kappa = -\frac{1}{2}\sqrt{\frac{\text{Scal}^g}{n(n-1)}}$. The alternative expressions for $n < 9$ easily occur using Theorem 3.1 and the given expression of γ^2 ; the restriction about n is taken due to the positivity of Scal^g . The relation (3.2) is a simple consequence of the equations $\nabla_X^c \varphi_0 = 0$ and $\nabla_X^g \varphi_0 = \frac{3\gamma}{4n}X \cdot \varphi_0$. Finally, for the action of the 4-form $dT = 2\sigma_T$ on φ_0 , viewed as a symmetric endomorphism, we apply the integrability condition for ∇^c -parallel spinors. \square

In particular, we observe that on a compact connected Riemannian spin manifold (M^n, g, T) , real Killing spinors with Killing number $\kappa = \frac{3\gamma}{4n}$, ∇^c -parallel spinors lying on Σ_γ , or characteristic spinors lying in Σ_γ , if existent, are sharing a common property: *They are all eigenspinors of the Riemannian Dirac operator with the same eigenvalue $-\frac{3\gamma}{4} \neq 0$, where $0 \neq \gamma \in \text{Spec}(T)$ is a real T -eigenvalue.*

Example 3.3. Let (M^6, g, J) be a (strict) 6-dimensional nearly Kähler manifold, see Section 4.1 for more details and references. Such a manifold admits two ∇^c -parallel spinors φ^\pm lying in Σ_γ with $\gamma = \pm 2\|T\|$ [31]. The scalar curvature is given by $\text{Scal}^g = \frac{15}{2}\|T\|^2$ and this coincides with $\frac{9(n-1)}{4n}\gamma^2 = \frac{9(n-1)}{2(9-n)}\|T\|^2$. Therefore, by Proposition 3.2 the spinors φ^\pm are real Killing spinors with $\kappa = 3\gamma/4n = \pm\|T\|/4$, a well-known result by [30, 38]. In particular, $dT \cdot \varphi^\pm = -\frac{1}{2}\text{Scal}^c \cdot \varphi^\pm = -\frac{3\gamma^2(n-3)}{2n}\varphi^\pm$, i.e. $dT \cdot \varphi^\pm = -3\|T\|^2\varphi^\pm$ (see also [31, Lem. 10.7]). Consider a 7-dimensional nearly parallel G_2 -manifold (more details are given in Section 4.2). There is a unique ∇^c -parallel spinor φ_0 with $\gamma = -\sqrt{7}\|T\|$ and the scalar curvature $\text{Scal}^g = \frac{27}{2}\|T\|^2$ coincides with $\frac{9(n-1)}{4n}\gamma^2 = \frac{9(n-1)}{2(9-n)}\|T\|^2$. Thus, φ_0 must be a Killing spinor with $\kappa = 3\gamma/4n = -\frac{3}{4\sqrt{7}}\|T\|$, which is well-known by [34, 31]. In particular, $dT \cdot \varphi_0 = -\frac{3\gamma^2(n-3)}{2n}\varphi_0$, i.e. $dT \cdot \varphi_0 = -6\|T\|^2\varphi_0$ (see also [31, Ex. 5.2]).

We proceed with some further properties of ∇^c -parallel spinors lying in Σ_γ for some $\gamma \neq 0$.

Lemma 3.4. *Assume that (M^n, g, T) is as in Proposition 3.2 and that $n = \dim M > 3$. Let $0 \neq \varphi_0 \in \Gamma(\Sigma)$ be a non-trivial spinor satisfying the equations given in (3.1). Then, $(X \wedge T) \cdot \varphi_0 \neq 0$ needs to hold for at least a vector field X .*

Proof. If $\nabla^c \varphi_0 = 0$, then using the identity $X \wedge T = X \lrcorner T + X \cdot T$ (see [15, p. 14] or [3, Appen. C]) we get

$$\nabla_X^g \varphi_0 = \frac{\gamma}{4}X \cdot \varphi_0 - \frac{1}{4}(X \wedge T) \cdot \varphi_0, \quad \forall X \in \Gamma(TM).$$

If $(X \wedge T) \cdot \varphi_0 = 0$ for any X , then $\nabla_X^g \varphi_0 = \frac{\gamma}{4}X \cdot \varphi_0$ for any X , which means that φ_0 is real Killing spinor with Killing number $\kappa = \gamma/4$. Thus the relation $\gamma^2 = 16\kappa^2$ needs to be true. We prove that this is contradiction. Indeed, based on the above discussion and using the values $\mp \frac{1}{2}\sqrt{\frac{\text{Scal}^g}{n(n-1)}}$ we get the relation $\gamma^2 = 16\kappa^2 = 16 \cdot \frac{\text{Scal}^g}{4n(n-1)} = \frac{4\text{Scal}^g}{n(n-1)}$. But then, by Theorem 3.1 it follows that $(1 - \frac{8}{n(n-1)})\text{Scal}^g = -\frac{1}{2}\|T\|^2$,

which is a contradiction for any $n > 3$, since $\text{Scal}^g > 0$ and $\|T\|^2 > 0$ ($T \neq 0$ by assumption). Finally, another direct argument applies. If φ_0 is a Killing spinor with $\kappa = \gamma/4$, then it must be an eigenspinor of the Riemannian Dirac operator D^g with eigenvalue $-n\gamma/4$. However, φ_0 is ∇^c -parallel and belongs to Σ_γ , hence it is also $D^g(\varphi_0) = -\frac{3\gamma}{4}\varphi_0$. Thus, for $n > 3$ this leads to a contradiction. \square

Of course, the Clifford multiplication of the 4-form $(X \wedge T)$ with a ∇^c -parallel spinor φ does not need to be an element of the subbundle $\{X \cdot \varphi : X \in TM\} \subset \Sigma$. In dimensions 6 and 7, for nearly Kähler manifolds and nearly parallel G_2 -manifolds, respectively, one can prove that there exists an appropriate constant c that makes the equation $(X \wedge T) \cdot \varphi = cX \cdot \varphi$ true for any vector field X and this corresponds to the deeper one-to-one correspondence between ∇^c -parallel spinors and Riemannian Killing spinors [31].

Lemma 3.5. *Let $0 \neq \gamma \in \text{Spec}(T)$ be a real non-zero T -eigenvalue. Then, a real Killing spinor $\varphi \in \mathcal{K}(M^n, g)_{\frac{3\gamma}{4n}}$ is characteristic i.e. $D^c(\varphi) = 0$, if and only if $\varphi \in \Sigma_\gamma$.*

Proof. Consider the Dirac operator $D^c \equiv D^{1/4}$ associated to the characteristic connection. Then $D^c(\varphi) = D^g(\varphi) + \frac{3}{4}T \cdot \varphi = -\frac{3\gamma}{4}\varphi + \frac{3}{4}T \cdot \varphi$, and the same occurs using the fact that $\nabla_X^c \varphi = \frac{1}{4}(X \lrcorner T) \cdot \varphi + \frac{3\gamma}{4n}X \cdot \varphi$:

$$\begin{aligned} D^c(\varphi) &= \sum_i e_i \cdot \nabla_{e_i}^c \varphi = \sum_i e_i \cdot \left\{ \frac{1}{4}(e_i \lrcorner T) \cdot \varphi + \frac{3\gamma}{4n}e_i \cdot \varphi \right\} = \frac{1}{4} \sum_i e_i \cdot (e_i \lrcorner T) \cdot \varphi + \frac{3\gamma}{4n} \sum_i e_i \cdot e_i \cdot \varphi \\ &= \frac{3}{4}T \cdot \varphi - \frac{3\gamma}{4}\varphi = \frac{3}{4}(T \cdot \varphi - \gamma\varphi). \end{aligned}$$

\square

Although on a compact Riemannian manifold with constant scalar curvature given by $\text{Scal}^g = \frac{9(n-1)}{4n}\gamma^2$ for some non-zero real eigenvalue $\gamma \in \text{Spec}(T)$, any existent ∇^c -parallel spinor $\varphi \in \Sigma_\gamma$ must be also a Killing spinor field, conversely we cannot claim that a Killing spinor with $\kappa = 3\gamma/4n$ is ∇^c -parallel, even if it is characteristic. From now on, we agree to denote by $\text{Ker}(\nabla^c)$ and $\text{Ker}(D^c)$ the following sets:

$$\text{Ker}(\nabla^c) := \{\varphi \in \Gamma(\Sigma_\gamma) \subset \Gamma(\Sigma) : \nabla^c \varphi \equiv 0\}, \quad \text{Ker}(D^c) := \{\varphi \in \Gamma(\Sigma_\gamma) \subset \Gamma(\Sigma) : D^c(\varphi) = 0\}.$$

With the aim to construct the desired one-to-one correspondence between spinor fields in $\text{Ker}(\nabla^c)$ and $\text{Ker}(D^c)$, it is necessary the eigenvalues of the endomorphism $dT + \frac{1}{2}\text{Scal}^c$ on Σ to be non-negative; this is a classical result of Th. Friedrich and S. Ivanov [31, Thm. 3.4]. In particular, for a Killing spinor $\varphi \in \mathcal{K}(M, g)_{\frac{3\gamma}{4n}}$ with $\varphi \in \Sigma_\gamma$ the relation $D^c(T \cdot \varphi) + T \cdot D^c(\varphi) = -2 \sum_i (e_i \lrcorner T) \cdot \nabla_{e_i}^c \varphi$ implies that $\sum_i (e_i \lrcorner T) \cdot \nabla_{e_i}^c \varphi = 0$ (see [31, Thm. 3.3]). Thus, integrating the generalized Schrödinger-Lichnerowicz formula associated to D^c , it follows that

$$0 = \int_M \left[\|\nabla^c \varphi\|^2 + \frac{1}{2} \langle dT \cdot \varphi + \frac{\text{Scal}^c}{2} \varphi, \varphi \rangle \right].$$

Combining now with Lemma 3.5 and Proposition 3.2, one deduces

Theorem 3.6. *Let (M^n, g, T) be compact connected Riemannian spin manifold (M^n, g, T) , with $\nabla^c T = 0$ and positive scalar curvature given by $\text{Scal}^g = \frac{9(n-1)}{4n}\gamma^2$ for some constant $0 \neq \gamma \in \text{Spec}(T)$. If the symmetric endomorphism $dT + \frac{1}{2} \left[\frac{9(n-1)}{4n}\gamma^2 - \frac{3}{2}\|T\|^2 \right]$ acts on Σ with non-negative eigenvalues, then the following classes of spinors, if existent, coincide*

$$\text{Ker}(\nabla^c) \cong \text{Ker}(D^c) \cong \bigoplus_{\gamma \in \text{Spec}(T)} \left[\Gamma(\Sigma_\gamma) \cap \mathcal{K}(M^n, g)_{\frac{3\gamma}{4n}} \right].$$

3.3. A basic one-to-one correspondence. Next we extend the correspondence established in Theorem 3.6 to ∇^c -parallel twistor and Killing spinors with torsion. A special version of the result that we discuss below, namely for nearly Kähler manifolds in dimension 6 and only for a specific parameter $s = 5/12$, is known by [3, Thm. 6.1]. Here, following a different method we present a more general version.

Notice that any twistor spinor with torsion with respect to $\nabla^c = \nabla^{1/4}$ which is characteristic $D^c(\varphi) = 0$, is in fact ∇^c -parallel. This also occurs after applying Lemma 2.2 for a characteristic twistor spinor; $\frac{1}{2}\text{Scal}^s \varphi = -4s(3-4s)\sigma_T \cdot \varphi$ and for $s = 1/4$ one has $\frac{1}{2}\text{Scal}^c \varphi = -2\sigma_T \cdot \varphi = -dT \cdot \varphi$ (notice that

in dimensions $n = 3, 4$ the 4-form σ_T is identically equal to zero). Thus, from now on we are mainly interested in twistors with torsion for some $s \neq 1/4$.

Theorem 3.7. *Let (M^n, g, T) be a compact connected Riemannian spin manifold with $\nabla^c T = 0$ and assume that $\varphi \in \Gamma(\Sigma)$ is a spinor field such that $\nabla^c \varphi = 0$, where $\nabla^c = \nabla^g + \frac{1}{2}T$ is the characteristic connection. Let $\gamma \in \mathbb{R} \setminus \{0\}$ be a non-zero real number. Then, the following conditions are equivalent:*

- (a) $\varphi \in \Gamma(\Sigma_\gamma) \cap \text{Ker}(P^s) := \text{Ker}(P^s|_{\Sigma_\gamma})$ with respect to the family $\{\nabla^s : s \in \mathbb{R} \setminus \{1/4\}\}$,
- (b) $\varphi \in \mathcal{K}^s(M, g)_\zeta$ with respect to the family $\{\nabla^s : s \in \mathbb{R} \setminus \{0, 1/4\}\}$ with $\zeta = 3(1 - 4s)\gamma/4n$,
- (c) $\varphi \in \mathcal{K}(M, g)_\kappa$ with $\kappa = 3\gamma/4n$.

Proof. By definition, it is $\nabla_X^c \varphi = \nabla_X^g \varphi + \frac{1}{4}(X \lrcorner T) \cdot \varphi$, hence we write

$$(3.3) \quad \nabla_X^s \varphi = \nabla_X^c \varphi + \frac{4s-1}{4}(X \lrcorner T) \cdot \varphi, \quad \forall X \in \Gamma(TM), \varphi \in \Gamma(\Sigma).$$

Because $D^c \equiv D^{1/4} = D^g + \frac{3}{4}T$, we also conclude that

$$D^s(\varphi) = D^g(\varphi) + 3sT \cdot \varphi = D^c(\varphi) + \frac{3(4s-1)}{4}T \cdot \varphi.$$

Then, the twistor spinor equation with respect to ∇^s can be expressed by

$$\nabla_X^c \varphi + \frac{4s-1}{4}(X \lrcorner T) \cdot \varphi + \frac{1}{n}X \cdot \{D^c(\varphi) + \frac{3(4s-1)}{4}T \cdot \varphi\} = 0, \quad \forall X \in \Gamma(TM).$$

Assume now that $\varphi \in \Gamma(\Sigma)$ is a twistor spinor with respect to ∇^s which verifies the equations (3.1). Then, for $s \neq \frac{1}{4}$, we conclude that the twistor equation is equivalent to the relation (3.2), namely

$$(X \lrcorner T) \cdot \varphi = -\frac{3\gamma}{n}X \cdot \varphi,$$

Thus, using (3.3) we can easily see that φ is a Killing spinor with torsion with respect to the family $\{\nabla^s : s \neq 0, \frac{1}{4}\}$ with Killing number $\zeta := 3(1 - 4s)\gamma/4n$ and moreover a Riemannian Killing spinor with Killing number $\kappa = 3\gamma/4n$:

$$\nabla_X^s \varphi = -\frac{3\gamma(4s-1)}{4n}X \cdot \varphi, \quad \text{and} \quad \nabla_X^g \varphi = \frac{3\gamma}{4n}X \cdot \varphi.$$

This proves the one direction $(a) \Rightarrow (b) \Rightarrow (c)$. Consider now some $\varphi \in \mathcal{K}^s(M, g)_\zeta$ with $s \neq 0, \frac{1}{4}$ and $\zeta := \frac{3\gamma(1-4s)}{4n}$ for some real parameter $\gamma \neq 0$. Under the additional assumption $\nabla^c \varphi = 0$, we will show that φ is an eigenspinor of T , i.e. $T \cdot \varphi = \gamma\varphi$. Indeed, any Killing spinor field with torsion is also a twistor spinor with torsion, hence the twistor equation yields the relation

$$\frac{4s-1}{4}(X \lrcorner T) \cdot \varphi + \frac{3(4s-1)}{4n}X \cdot T \cdot \varphi = 0. \quad (\#)$$

On the other hand, for any $s \neq 0, \frac{1}{4}$ it holds that $\nabla_X^s \varphi = \zeta X \cdot \varphi$ with $\zeta := \frac{3\gamma(1-4s)}{4n} \neq 0$. Due to (3.3) we finally get

$$\frac{4s-1}{4}(X \lrcorner T) \cdot \varphi = \zeta X \cdot \varphi,$$

for any vector field X . Inserting this in $(\#)$ we see that

$$X \cdot (\zeta\varphi + \frac{3(4s-1)}{4n}T \cdot \varphi) = 0,$$

and our claim follows. This shows the direction $(b) \Rightarrow (a)$ and remains to prove also that (c) implies (b) . Assume that $\varphi \in \mathcal{K}(M, g)_\kappa$ is a real Killing spinor with Killing number $\kappa := 3\gamma/4n$ and such that $\nabla_X^c \varphi = 0$ for any vector field X . Then, $\nabla_X^g \varphi = -\frac{1}{4}(X \lrcorner T) \cdot \varphi = \frac{3\gamma}{4n}X \cdot \varphi$, and using (3.3) we complete the proof. \square

Remarks: Choosing one of the conditions (b) or (c) in Theorem 3.7 for some real constant $\gamma \neq 0$, we see that the relation $T \cdot \varphi = \gamma\varphi$ follows by the twistor equation. We also remark that one can start with a triple (M^n, g, T) with $\nabla^c T = 0$, endowed with a real Killing spinor $\varphi \in \mathcal{K}(M^n, g)_{\frac{3\gamma}{4n}} \cap \Gamma(\Sigma_\gamma)$

for some $0 \neq \gamma \in \text{Spec}(T)$, and similarly prove that the conditions $\varphi \in \text{Ker}(\nabla^c)$, $\varphi \in \mathcal{K}^s(M, g)_\zeta$ with $\zeta = 3\gamma(1-4s)/4n$ and $\varphi \in \text{Ker}(P^s|_{\Sigma_\gamma})$ are equivalent one to each other.

A first immediate corollary of Theorem 3.7 is the following one:

Corollary 3.8. *Let (M^n, g, T) a triple with $\nabla^c T = 0$, carrying a ∇^c -parallel spinor field $0 \neq \varphi$ satisfying one of the conditions (a), (b) or (c) in Theorem 3.7. Then, the following holds for any vector field X*

$$(X \wedge T) \cdot \varphi = \frac{(n-3)\gamma}{n} X \cdot \varphi.$$

Conversely, if $\varphi \in \Sigma_\gamma$ is a ∇^c -parallel spinor satisfying the previous relation for any $X \in \Gamma(TM)$ and for some real $0 \neq \gamma \in \text{Spec}(T)$, then the conditions given in Theorem 3.7 must hold, in particular φ is a real Killing spinor with respect to g with Killing number $\kappa = \frac{3\gamma}{4n}$. Finally, for $n = 3$ it is $(X \wedge T) \cdot \varphi = 0$ identically, i.e. $(X \lrcorner T) \cdot \varphi = -\gamma X \cdot \varphi = -X \cdot T \cdot \varphi$.

Proof. The key ingredient of the proof is encoded in (3.2). Given a ∇^c -parallel spinor satisfying one of the conditions in Theorem 3.7, then $\varphi \in \Sigma_\gamma$ for some $\gamma \neq 0$ and the relation (3.2) needs to hold. Thus, the result easily follows due to the identity $X \wedge T = X \lrcorner T + X \cdot T$. For $n = 3$ we get $(X \wedge T) = 0$ and so $X \lrcorner T = -X \cdot T$ (as it should be for dimensional reasons). \square

Now, similar with Lemma 3.5 we observe that

Lemma 3.9. *Let $0 \neq \gamma \in \text{Spec}(T)$ be a non-zero real T -eigenvalue. Then the following hold:*

(a) *A Killing spinor with torsion $\varphi \in \mathcal{K}^s(M, g)_{\frac{3\gamma(1-4s)}{4n}}$ for some $s \neq 0, 1/4$ is characteristic, if and only $\varphi \in \Sigma_\gamma$.*

(b) *A twistor spinor with torsion $\varphi \in \text{Ker}(P^s|_{\Sigma_\gamma})$ for some $s \neq 0, 1/4$ is characteristic, if and only if φ is a D^s -eigenspinor with eigenvalue $-\frac{3\gamma(1-4s)}{4}$ i.e. $\varphi \in \mathcal{K}^s(M, g)_{\frac{3\gamma(1-4s)}{4n}} \cap \Gamma(\Sigma_\gamma)$. In particular, for $s = 0$, a twistor spinor $\varphi \in \text{Ker}(P^g|_{\Sigma_\gamma})$ is characteristic, if and only if $D^g(\varphi) = -\frac{3\gamma}{4}\varphi$, i.e. $\varphi \in \mathcal{K}(M, g)_{\frac{3\gamma}{4n}} \cap \Gamma(\Sigma_\gamma)$.*

Proof. (a) The first claim is a simple consequence of $D^s(\varphi) = D^c(\varphi) + \frac{3(4s-1)}{4}T \cdot \varphi$ and the fact that φ is an eigenspinor of D^s with eigenvalue $-\frac{3\gamma(1-4s)}{4}$.

(b) Consider a twistor spinor with torsion $\varphi \in \text{Ker}(P^s)$ for some $s \neq 0, 1/4$ such that $T \cdot \varphi = \gamma\varphi$. We write the twistor equation as

$$\nabla_X^s \varphi + \frac{1}{n} X \cdot D^c(\varphi) + \frac{3(4s-1)}{4n} X \cdot T \cdot \varphi = 0,$$

which is equivalent to

$$\frac{1}{n} X \cdot (D^c - D^s)(\varphi) = -\frac{3\gamma(4s-1)}{4n} X \cdot \varphi.$$

Since the latter equation holds for any vector field X , we easily conclude. A more direct way is given as follows: Suppose that $\varphi \in \text{Ker}(P^s|_{\Sigma_\gamma})$ is in addition characteristic, i.e. $D^c(\varphi) = 0$. Then

$$\nabla_X^s \varphi = -\frac{1}{n} X \cdot D^s(\varphi) = -\frac{1}{n} X \cdot [D^c(\varphi) + \frac{3(4s-1)}{4}T \cdot \varphi] = \frac{3\gamma(1-4s)}{4n} X \cdot \varphi,$$

i.e. $\varphi \in \mathcal{K}^s(M, g)_{\frac{3\gamma(1-4s)}{4n}}$ and the converse is obvious. Similarly for $s = 0$. \square

Combining Theorem 3.6 with Theorem 3.7 and Lemma 3.9, we take the following extension.

Corollary 3.10. *Let (M^n, g, T) be compact connected Riemannian spin manifold (M^n, g, T) , with $\nabla^c T = 0$ and positive scalar curvature given by $\text{Scal}^g = \frac{9(n-1)\gamma^2}{4n}$ for some constant $0 \neq \gamma \in \text{Spec}(T)$. If the symmetric endomorphism $dT + \frac{1}{2} \left[\frac{9(n-1)}{4n} \gamma^2 - \frac{3}{2} \|T\|^2 \right]$ acts on Σ with non-negative eigenvalues, then the following classes of spinors, if existent, coincide*

$$\text{Ker}(\nabla^c) \cong \bigoplus_{\gamma \in \text{Spec}(T)} \left[\Gamma(\Sigma_\gamma) \cap \mathcal{K}(M, g)_{\frac{3\gamma}{4n}} \right] \cong \bigoplus_{\gamma \in \text{Spec}(T)} \left[\Gamma(\Sigma_\gamma) \cap \mathcal{K}^s(M, g)_{\frac{3\gamma(1-4s)\gamma}{4n}} \right] \cong \bigoplus_{\gamma \in \text{Spec}(T)} \left[\text{Ker}(P^s|_{\Sigma_\gamma}) \cap \text{Ker}(D^c) \right].$$

Here, the parameter s takes values in $\mathbb{R} \setminus \{0, 1/4\}$ for the third set, and for the last set it is $s \in \mathbb{R} \setminus \{1/4\}$.

4. EXAMPLES

The most representative classes of special structures for which Theorems 3.6, 3.7 and Corollaries 3.8, 3.10 make sense, are 6-dimensional nearly Kähler manifolds and 7-dimensional nearly parallel G₂-manifolds. Of course the same holds for Proposition 3.2, which was the starting point of this theoretical approach. Let us describe these special structures in some detail.

4.1. Nearly Kähler manifolds and their spinorial properties. A nearly Kähler manifold is an almost Hermitian manifold endowed with an almost complex structure J such that $(\nabla_X^g J)X = 0$. Next we need to recall basic results from [31, 30, 38, 15, 7], where we refer for more details and proofs. In dimension 6 strict nearly Kähler manifolds (M, g, J) are very special; they are spin, the first Chern class vanishes $c_1(M^6, J) = 0$ and g is an Einstein metric. In the homogeneous case, 6-dimensional nearly Kähler manifolds are exhausted by the 3-symmetric spaces $\mathbb{S}^6 = \mathrm{G}_2 / \mathrm{SU}_3$, $\mathbb{CP}^3 = \mathrm{SO}_5 / \mathrm{U}_2 = \mathrm{Sp}_2 / (\mathrm{Sp}_1 \times \mathrm{U}_1)$, $\mathbb{F}_{1,2} = \mathrm{SU}_3 / T_{\max}$ and $\mathrm{S}^3 \times \mathrm{S}^3 = \mathrm{SU}_2 \times \mathrm{SU}_2$, endowed with a naturally reductive (Einstein) metric [21]. Together with the standard spheres S^{2m} , these spaces exhaust all even-dimensional Riemannian manifolds admitting real Killing spinors. Notice that recently in [23], a *locally homogeneous* nearly Kähler manifold of the form $M = \tilde{M}/\Gamma$ was described, where $\tilde{M} = \mathrm{S}^3 \times \mathrm{S}^3$ and Γ is any finite subgroup of $\mathrm{SU}_2 \times \mathrm{SU}_2$.

Now, any nearly Kähler manifold admits a characteristic connection ∇^c with parallel skew-torsion, given by $T(X, Y) := (\nabla_X^g J)JY$ (Gray connection) [31, Thm. 10.1]. In particular, in dimension 6 there exists a positive constant $\tau_0 \neq 0$ such that (see [31, 7, 11])

$$\|T\|^2 = 2\tau_0, \quad \mathrm{Scal}^g = 15\tau_0, \quad \mathrm{Scal}^c = 12\tau_0, \quad \mathrm{Ric}^g = \frac{5}{2}\tau_0 \mathrm{Id}.$$

Notice that working with an even dimensional manifold M^{2m} of constant positive scalar curvature, the spinor bundle splits $\Sigma = \Sigma^+ \oplus \Sigma^-$ and there is a bijection between the subbundles

$$E_{\pm} = \{\varphi \in \Gamma(\Sigma) : \nabla_X^g \varphi \pm \kappa X \cdot \varphi = 0, \forall X \in \Gamma(TM)\},$$

given by the map $\varphi^+ := \psi^+ + \psi^- \mapsto \varphi^- := \psi^+ - \psi^-$ for some $\varphi^+ \in E_+$ and $\psi^{\pm} \in \Sigma^{\pm}$, where κ is given as in the proof of Proposition 3.2. If $M^{2m} \neq \mathrm{S}^{2m}$, then $\dim_{\mathbb{C}} E_{\pm} \leq 2^{m-1} = \dim_{\mathbb{C}} \Delta_{2m}^{\pm}$ while for the standard spheres S^{2m} we have the characterization $\dim_{\mathbb{C}} E_{\pm} = 2^m$ [25]. For a 6-dimensional nearly Kähler manifold it is well-known that there are two Riemannian Killing spinors φ^{\pm} , i.e. $\dim_{\mathbb{C}} E_+ = \dim_{\mathbb{C}} E_- = 1$ [30, 38]. Moreover, φ^{\pm} are T -eigenspinors with eigenvalues $\gamma := \pm 2\|T\|$ and exhaust all ∇^c -parallel spinors [31, p. 333].

Theorem 4.1. *On a 6-dimensional nearly Kähler manifold (M^6, g, J) endowed with its characteristic connection ∇^c , the following classes of spinor fields coincide:*

- (1) *TsT with respect to the family $\{\nabla^s : s \in \mathbb{R} \setminus \{1/4\}\}$, lying in $\Sigma_{\pm 2\|T\|}$,*
- (2) *KsT with respect to the family $\{\nabla^s : s \in \mathbb{R} \setminus \{0, 1/4\}\}$, with $\zeta := \mp \frac{(4s-1)}{4}\|T\|$,*
- (3) *Riemannian Killing spinors,*
- (4) *∇^c -parallel spinors.*

Proof. Although one can immediately apply Theorem 3.7, let us follow a bit different approach. Assume that φ^{\pm} are TsT with respect the family ∇^s , for some $s \neq 1/4$. For $n = 6$ and for $s = 5/12$ it is known [3, Cor. 6.1] that an element $\varphi \in \mathrm{Ker}(P^{5/12}|_{\Sigma_{\gamma}})$ satisfies the equation

$$\nabla_X^c \varphi - \frac{1}{18} \left(\gamma + 2 \frac{\|T\|^2}{\gamma} \right) X \cdot \varphi + \frac{1}{6} (X \wedge T) \cdot \varphi = 0,$$

and this is also the Killing equation with torsion. Now, $\varphi^{\pm} \in \Sigma_{\pm 2\|T\|}$ are TsT for some $s \neq 1/4$. Thus, one may assume without loss of generality that $s = 5/12$ and then φ^{\pm} ought to satisfy the previous equation as well. Because $\nabla^c \varphi^{\pm} = 0$, this finally reduces to $\pm \|T\| X \cdot \varphi^{\pm} = (X \wedge T) \cdot \varphi^{\pm}$ or equivalently

$$(4.1) \quad (X \lrcorner T) \cdot \varphi^{\pm} = \pm \|T\| X \cdot \varphi^{\pm} \mp 2\|T\| X \cdot \varphi^{\pm} = \mp \|T\| X \cdot \varphi^{\pm}.$$

Of course, this is exactly what one gets after running our twistor equation, i.e. apply (3.2) for $\gamma = \pm 2\|T\|$. Hence, a simple application of (3.3) yields the result: $\nabla_X^s \varphi^{\pm} = \frac{(4s-1)}{4} (X \lrcorner T) \cdot \varphi^{\pm} = \mp \frac{(4s-1)}{4} \|T\| X \cdot \varphi^{\pm}$, and similar for the Levi-Civita connection. \square

Remarks: In [3, Thm. 6.1] the Killing number with torsion is given by $\mp \frac{\|T\|}{6}$ and this coincides with the statement of Theorem 4.1 for $s = 5/12$. In this way, we generalise this result by extending the correspondence to any real number $s \neq 0, 1/4$. Moreover, and relative to Corollary 3.8, notice that any vector field X satisfies $(X \wedge T) \cdot \varphi^\pm \neq 0$, in particular $\pm \|T\| X \cdot \varphi^\pm = (X \wedge T) \cdot \varphi^\pm$.

4.2. Nearly parallel G_2 -manifolds and their spinorial properties. A 7-dimensional oriented Riemannian manifold (M^7, g) is called a G_2 -manifold whenever the structure group of its frame bundle is contained in $G_2 \subset SO_7$. The existence of such a reduction amounts to the existence of a generic 3-form ω . Since G_2 preserves ω and the same time a unit spinor $\varphi_0 \in \Delta_7$, they ought to induce the same data, namely (we refer to [19, 2, 31, 29] for details on G_2 -structures)

$$\omega(X, Y, Z) := (X \cdot Y \cdot Z \cdot \varphi_0, \varphi_0).$$

The GL_7 -orbit of ω in $\bigwedge^3(\mathbb{R}^7)$, is an open set which we shall denote by $\bigwedge_+^3(\mathbb{R}^7)$. Sections of the bundle $\bigwedge_+^3 TM := \cup_{x \in M} \bigwedge_+^3(T_x^* M)$ are called stable 3-forms and it is well-known that there is a bijection between G_2 -structures on M and sections $\omega \in \Gamma(\bigwedge_+^3 TM) := \Omega_+^3(M)$ [34]. Given such a 3-form it determines a Riemannian metric and induces an orientation on M . A nearly parallel G_2 -structure on M^7 is a G_2 -structure $\omega \in \Omega_+^3(M)$ satisfying the differential equation $d\omega = -\tau_0 * \omega$ for some real constant $\tau_0 \neq 0$. The existence of such a structure is equivalent with the existence of a spin structure carrying a real Killing spinor [34, 33].

A nearly parallel G_2 -manifold admits a unique characteristic connection ∇^c with parallel skew-torsion T given by $T := \frac{1}{6}(d\omega, * \omega) \cdot \omega$ [31, Cor. 4.9]. The positive real number τ_0 links T and ω , in particular it holds that (see [31, 7, 11]):

$$T = -\frac{\tau_0}{6}\omega, \quad \|T\|^2 = \frac{7}{36}\tau_0^2, \quad \text{Ric}^g = \frac{3}{8}\tau_0^2 \text{Id}, \quad \text{Scal}^g = \frac{21}{8}\tau_0^2, \quad \text{Scal}^c = \frac{7}{3}\tau_0^2.$$

The connection ∇^c admits a unique parallel spinor field φ_0 of length one such that (here we work with 3-form ω such that $\omega \cdot \varphi_0 = 7\varphi_0$, see [4, Lem. 2.3])

$$(4.2) \quad T \cdot \varphi_0 = -\frac{7\tau_0}{6}\varphi_0 = -\sqrt{7}\|T\|\varphi_0.$$

In fact, φ_0 is a real Killing spinor [34, 31], with Killing number $\kappa = -\frac{1}{2}\sqrt{\frac{\text{Scal}^g}{n(n-1)}} = -\frac{\tau_0}{8} = -\frac{3}{4\sqrt{7}}\|T\|$ and an eigenspinor of D^g with eigenvalue $\frac{1}{2}\sqrt{\frac{n \text{Scal}^g}{n-1}} = \frac{7\tau_0}{8} = \frac{3\sqrt{7}}{4}\|T\|$. This can be seen also as follows:

$$D^g \varphi_0 = D^c \varphi_0 - \frac{3}{4}T \cdot \varphi_0 = -\frac{3}{4}T \cdot \varphi_0 = \frac{3\sqrt{7}}{4}\|T\|\varphi_0.$$

Therefore, after applying Theorem 3.7 one deduces that

Theorem 4.2. *On a nearly-parallel G_2 -manifold (M^7, g, ω) endowed with its characteristic connection ∇^c , the following classes of spinor fields coincide:*

- (1) TsT with respect to the family $\{\nabla^s : s \in \mathbb{R} \setminus \{1/4\}\}$, lying in $\Sigma_{-\frac{7\tau_0}{6}} \equiv \Sigma_{-\sqrt{7}\|T\|}$,
- (2) KsT with respect to the family $\{\nabla^s : s \in \mathbb{R} \setminus \{0, 1/4\}\}$, with $\zeta := \frac{(4s-1)\tau_0}{8} = \frac{3(4s-1)\|T\|}{4\sqrt{7}}$,
- (3) Riemannian Killing spinors,
- (4) ∇^c -parallel spinors.

Proof. We shall present a proof using slightly different arguments. Given some global non-trivial spinor φ_0 of the (real) spin representation $\Delta_7 \cong \mathbb{R}^8$, one has the decomposition $\Delta_7 = \mathbb{R}\varphi_0 \oplus \{X \cdot \varphi_0 : X \in \mathbb{R}^7\}$. On a nearly parallel G_2 -manifold, the spinor φ_0 is the unique ∇^c -parallel spinor [31, 7]. The induced 3-form $\omega(X, Y, Z) = (X \cdot Y \cdot Z \cdot \varphi_0, \varphi_0)$ is such that $\|\omega\| = \sqrt{7}$, $\omega \cdot \varphi_0 = 7\varphi_0$, see for example [4]. Now, the space $\mathfrak{so}(7) \cong \Lambda^2(\mathbb{R}^7)$ decomposes under the G_2 -action as $\Lambda_7^2 \oplus \mathfrak{g}_2$, where $\Lambda_7^2 = \{X \lrcorner \omega : X \in \mathbb{R}^7\}$. Then, one deduces that $(X \lrcorner \omega) \cdot \varphi_0$ must be proportional to $X \cdot \varphi_0$, in particular (see [4, Lem. 2.3] or [10, p. 33])

$$(X \lrcorner \omega) \cdot \varphi_0 = -3X \cdot \varphi_0, \quad \forall X \in \Gamma(TM).$$

Then, a simple combination with $T = -\frac{\tau_0}{6}\omega$ gives rise to

$$(4.3) \quad (X \lrcorner T) \cdot \varphi_0 = \frac{\tau_0}{2} X \cdot \varphi_0 = \frac{3\|T\|}{\sqrt{7}} X \cdot \varphi_0.$$

Of course, the same result occurs after applying our twistor equation for φ_0 and for some $s \neq 1/4$, see (3.2) and the proof of Theorem 3.7. Due to ∇^c -parallelism of φ_0 we finally conclude that

$$\begin{aligned} \nabla_X^s \varphi_0 &= \nabla_X^c \varphi_0 + \frac{(4s-1)}{4} (X \lrcorner T) \cdot \varphi_0 = \frac{(4s-1)}{4} (X \lrcorner T) \cdot \varphi_0 \\ &= \frac{(4s-1)\tau_0}{8} X \cdot \varphi_0 = \frac{3(4s-1)\|T\|}{4\sqrt{7}} X \cdot \varphi_0. \end{aligned}$$

Thus, for any $s \neq 0, 1/4$ it is $\varphi_0 \in \mathcal{K}(M^7, g)_\zeta$ with $\zeta := \frac{(4s-1)\tau_0}{8}$ and moreover $\varphi_0 \in \mathcal{K}(M^7, g)_\kappa$ with $\kappa = -\frac{\tau_0}{8} = -\frac{3\|T\|}{4\sqrt{7}}$. The opposite direction is based again in (4.3), which also occurs by the Riemannian Killing spinor equation $\nabla_X^g \varphi_0 = -\frac{1}{4} (X \lrcorner T) \cdot \varphi_0 = -\frac{3\|T\|}{4\sqrt{7}} X \cdot \varphi_0$. Hence, using (3.3) it follows that φ_0 is also a KsT for any $s \neq 0, 1/4$, and finally by the twistor equation we get that $\varphi_0 \in \Sigma_{-\sqrt{7}\|T\|}$. \square

4.3. An explicit example. Let us describe an explicit example, namely the space $B^7 = \mathrm{SO}_5 / \mathrm{SO}_3^{\mathrm{ir}}$. M. Berger proved that this is a space of positive sectional curvature.

Consider the space $\mathrm{Sym}_0^2(\mathbb{R}^3)$ of (3×3) symmetric traceless matrices; we identify $\mathrm{Sym}_0^2(\mathbb{R}^3) \cong \mathbb{R}^5$ by viewing any vector $(x_1, \dots, x_5)^t$ in \mathbb{R}^5 as a real matrix A of the form

$$A = \begin{bmatrix} \frac{x_5}{\sqrt{3}} + x_1 & x_2 & x_3 \\ x_2 & \frac{x_5}{\sqrt{3}} - x_1 & x_4 \\ x_3 & x_4 & -\frac{2x_5}{\sqrt{3}} \end{bmatrix} \in \mathrm{Sym}_0^2(\mathbb{R}^3).$$

The Lie group SO_3 acts on $\mathrm{Sym}_0^2(\mathbb{R}^3) \cong \mathbb{R}^5$ by conjugation $\iota(h)A = hAh^t$. This defines the unique 5-dimensional SO_3 -irreducible representation and an embedding of SO_3 inside SO_5 , which we shall denote by $\mathrm{SO}_3^{\mathrm{ir}} := \iota(\mathrm{SO}_3) \subset \mathrm{SO}_5$. For the Lie algebra $\mathfrak{so}(3) \subset \mathfrak{so}(5)$ we fix the standard basis, i.e. $\mathfrak{so}(3) = \mathrm{span}\{y_1 := E_{2,3}, y_2 := -E_{1,3}, y_3 := E_{1,2}\}$ such that $[y_1, y_2] = y_3$, $[y_2, y_3] = y_1$ and $[y_3, y_1] = y_2$, where $E_{i,j}$ denote the endomorphisms mapping e_i to e_j , e_j to $-e_i$ and everything else to zero. The embedding $\mathfrak{so}(3)_{\mathrm{ir}} \subset \mathfrak{so}(5)$ is explicitly given by

$$y_1 \mapsto \iota_*(y_1) = \sqrt{3}E_{1,5} - E_{2,5} + E_{3,4}, \quad y_2 \mapsto \iota_*(y_2) = -\sqrt{3}E_{1,4} - E_{2,4} - E_{3,5}, \quad y_3 \mapsto \iota_*(y_3) = 2E_{2,3} + E_{4,5}.$$

These matrices have length equal to $\sqrt{5}$ and define an orthogonal basis of $\mathfrak{so}(3)_{\mathrm{ir}} := \iota_*(\mathfrak{so}(3))$ with respect to the scalar product $(A, B) = -(1/2) \mathrm{tr} AB$, i.e. $\mathfrak{so}(3)_{\mathrm{ir}} = \mathrm{span}\{\iota_*(y_1), \iota_*(y_2), \iota_*(y_3)\}$.

Let $\mathfrak{so}(5) = \mathfrak{so}(3)_{\mathrm{ir}} \oplus \mathfrak{m}$ be a reductive decomposition and let us denote by $\langle \cdot, \cdot \rangle = (\cdot, \cdot)_{\mathfrak{m} \times \mathfrak{m}}$ the normal metric induced by $(\cdot, \cdot) : \mathfrak{so}(5) \times \mathfrak{so}(5) \rightarrow \mathbb{R}$. We identify $\mathfrak{m} \cong \mathbb{R}^7$ with the imaginary octonions $\mathrm{Im}(\mathbb{O})$ and construct an orthonormal basis of \mathfrak{m} such that $[e_i, e_{i+1}]$ be a multiple of e_{i+3} , where the indices are permuted cyclically and translated modulo 7. For example, set $e_1 := -E_{1,2}$ and

$$\begin{aligned} e_2 &:= -E_{1,3}, & e_4 &:= \frac{1}{\sqrt{5}}(E_{2,3} - 2E_{4,5}), & e_6 &:= \frac{1}{2}(E_{1,5} - \sqrt{3}E_{3,4}), \\ e_3 &:= -\frac{1}{2}(E_{1,4} - \sqrt{3}E_{3,5}), & e_5 &:= \frac{2}{\sqrt{5}}(E_{2,5} + \frac{1}{4}E_{3,4} + \frac{\sqrt{3}}{4}E_{1,5}), & e_7 &:= \frac{2}{\sqrt{5}}(E_{2,4} - \frac{1}{4}E_{3,5} - \frac{\sqrt{3}}{4}E_{1,4}). \end{aligned}$$

Then $[e_i, e_{i+1}] = ce_{i+3}$ with $c := 1/\sqrt{5}$ (see also [37] but be aware for another realization of $\mathfrak{so}(3)_{\mathrm{ir}} \subset \mathfrak{so}(5)$ and so another basis of \mathfrak{m}). Since both $\mathrm{SO}_3^{\mathrm{ir}}$ and G_2 preserving the splitting $\mathbb{O} = \mathbb{R} \oplus \mathrm{Im}(\mathbb{O})$, the G_2 -equivariant identification $\mathfrak{m} \cong \mathrm{Im}(\mathbb{O})$ induces a 2-fold cross product and hence an invariant G_2 -structure on B^7 . Thus, the image of $\mathrm{SO}_3^{\mathrm{ir}}$ via the isotropy representation lies inside G_2 . The isotropy representation coincides with the unique 7-dimensional irreducible representation of SO_3 , which is induced by the action of SO_3 on the space $\mathrm{Sym}_0^3(\mathbb{R}^3) \cong \mathbb{R}^7$ of trace-free 3-symmetric tensors on \mathbb{R}^3 .

Let us describe the 3-form ω associated to the G_2 -structure. It is well-known that the space of invariant 3-forms on B^7 contains the trivial summand with multiplicity one, i.e. $\mathbb{R} \subset \bigwedge^3(\mathfrak{m})^\mathfrak{k}$ with $\mathfrak{k} := \mathfrak{so}(3)_{\mathrm{ir}}$, see [19, 12]. Since B^7 is (strongly) isotropy irreducible, Schur's lemma ensures that this corresponds to

the torsion of the canonical connection ∇^1 with respect to the fixed reductive decomposition $\mathfrak{so}(5) = \mathfrak{so}(3)_{\text{ir}} \oplus \mathfrak{m}$, namely

$$T^1 = \sum_{i < j < k} T^1(e_i, e_j, e_k) e_{ijk} = -\frac{1}{\sqrt{5}}(e_{124} + e_{137} + e_{156} + e_{235} + e_{267} + e_{346} + e_{457}),$$

where we write $e_{i_1 \dots i_k}$ for the wedge product $e_{i_1} \wedge \dots \wedge e_{i_k} \in \bigwedge^k(\mathbb{R}^7)^*$. It is easy to see that $\|T^1\|^2 = \frac{1}{3} \sum_{i < j} \langle T^1(e_i, e_j), T^1(e_i, e_j) \rangle = \frac{21}{3} c^2 = 7/5$, hence let us set $\omega := -\sqrt{5} T^1$ such that $\|\omega\|^2 = 7$. Obviously $\omega \in \Omega_+^3$ and its Hodge dual is given by $*\omega = e_{1236} - e_{1257} - e_{1345} + e_{1467} + e_{2347} - e_{2456} - e_{3567}$.

Based on simple representation theory we deduce that ω induces a nearly parallel G_2 -structure. For example, the Hodge star operator $*$ allows us to identify $\bigwedge^3(\mathfrak{m})^{\mathfrak{k}} \cong \bigwedge^4(\mathfrak{m})^{\mathfrak{k}}$, as \mathfrak{k} -modules. The differential of a 3-form $\bigwedge^3(\mathfrak{m})^{\mathfrak{k}}$ is again \mathfrak{k} -invariant and the exterior differential $d : \bigwedge^3(\mathfrak{m})^{\mathfrak{k}} \rightarrow \bigwedge^4(\mathfrak{m})^{\mathfrak{k}}$ is an equivariant map. Hence $d\omega$ must be a multiple of the trivial summand in $\bigwedge^4(\mathfrak{m})^{\mathfrak{k}}$ which means that $d\omega$ and $*\omega$ must be proportional. In particular, we see that $dT^1 = -\frac{6}{5}(e_{1236} - e_{1257} - e_{1345} + e_{1467} + e_{2347} - e_{2456} - e_{3567})$ and since $d\omega = -\sqrt{5}dT^1$ it follows that $d\omega = \frac{6}{\sqrt{5}}*\omega$. Therefore $d\omega \neq 0$ (which is equivalent to say that ω is not parallel) and moreover $d*\omega = 0$, i.e. ω is co-calibrated. The co-differential vanishes $\delta\omega = 0$ (since $\delta T^1 = 0$) and hence the equation $d\omega = -8\kappa*\omega$ is equivalent to the Killing spinor equation $\nabla_X^g \varphi = \kappa X \cdot \varphi$, see [34, Prop. 3.12]. It follows that the coset $(\text{SO}_5 / \text{SO}_3^{\text{ir}}, \langle \cdot, \cdot \rangle, \omega)$ is a homogeneous nearly parallel G_2 -manifold with Killing number $\kappa := -\frac{3}{4\sqrt{5}}$ and constant scalar curvature $\text{Scal}^g = 4n(n-1)\kappa^2 = 189/10$. The corresponding Killing spinor field φ_0 is ∇^1 -parallel [31, Thm. 5.6], in fact φ_0 generates the space of all ∇^1 -parallel spinors. Because $\text{SO}_5 / \text{SO}_3^{\text{ir}}$ is normal homogeneous, it is $\mathfrak{g} = \hat{\mathfrak{g}} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ and φ_0 necessarily corresponds to a constant map $\varphi_0 : G \rightarrow \Delta_7$ [1, Thm. 4.2]. Thus, any $X \in \mathfrak{m}$ satisfies the equation $\nabla_X^g \varphi = \nabla_X^1 \varphi + \widetilde{\Lambda}^g(X) \varphi = \widetilde{\Lambda}^g(X) \varphi_0 = -\frac{3}{4\sqrt{5}} X \cdot \varphi_0$, where one describes the lift $\widetilde{\Lambda}^g : \mathfrak{m} \rightarrow \mathfrak{spin}(\mathfrak{m})$ by using the Nomizu map $\Lambda^g(X)Y = (1/2)[X, Y]_{\mathfrak{m}}$ of the Levi-Civita connection ∇^g and applying the rule $\mathfrak{so}(7) \ni E_{i,j} \mapsto (e_i \cdot e_j/2) \in \mathfrak{spin}(7)$. For the endomorphism $\Lambda^g : \mathfrak{m} \rightarrow \mathfrak{so}(7)$ we compute

$$\begin{aligned} \Lambda^g(e_1) &= \frac{c}{2}(E_{2,4} + E_{3,7} + E_{5,6}), & \Lambda^g(e_5) &= -\frac{c}{2}(E_{1,6} - E_{2,3} + E_{4,7}), \\ \Lambda^g(e_2) &= \frac{c}{2}(-E_{1,4} + E_{3,5} + E_{6,7}), & \Lambda^g(e_6) &= \frac{c}{2}(E_{1,5} - E_{2,7} + E_{3,4}), \\ \Lambda^g(e_3) &= -\frac{c}{2}(E_{1,7} + E_{2,5} - E_{4,6}), & \Lambda^g(e_7) &= \frac{c}{2}(E_{1,3} - E_{2,6} + E_{4,5}), \\ \Lambda^g(e_4) &= \frac{c}{2}(E_{1,2} - E_{3,6} + E_{5,7}), \end{aligned}$$

Then, the relation $\widetilde{\Lambda}^\alpha = (1 - \alpha)\widetilde{\Lambda}^g$ shows that φ_0 is also a non-trivial Killing spinor with torsion with Killing number $\zeta = \frac{-3(1-\alpha)}{4\sqrt{5}}$, for any $\alpha \neq 0, 1$. It remains to examine the eigenvalues of the T^1 -action on the spinor bundle $\Sigma = \text{SO}_5 \times_\rho \Delta_7$. The real Clifford algebra $\text{Cl}(\mathbb{R}^7)$ coincides with $M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$ and the spin representation Δ_7 is a real representation. The Clifford representation attains the matrix realization given in [34, p. 261] or [15, p. 96] and then, as an endomorphism of $\Delta_7 := \mathbb{R}^8$, the torsion form T^1 reads

$$T^1 = -\frac{1}{\sqrt{5}} \begin{pmatrix} 0 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 0 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 0 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 1 & 0 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & 0 \end{pmatrix} \in \text{End}(\Delta_7).$$

We see that there is a unique negative eigenvalue with multiplicity one, namely $\gamma := -7/\sqrt{5}$ and this is exactly the result described in Theorem 3.7 or 4.2:

$$\gamma := \frac{4n\kappa}{3} \Rightarrow \gamma = -\frac{4 \cdot 7 \cdot 3}{3 \cdot 4 \cdot \sqrt{5}} = -7/\sqrt{5} = -\sqrt{7}\|T^1\|.$$

We finish with a remark about the first eigenvalue of the cubic Dirac operator $\not{D} \equiv D^{1/3}$. According to [1, Thm. 3.3], the square of \not{D} is given by $\not{D}^2(\varphi) = \Omega_{\mathfrak{so}(5)}(\varphi) + \frac{1}{8} \text{Scal}^{1/3} \varphi + \frac{3}{4} \|T^{1/3}\|^2 \varphi$. A short computation shows that $\text{Scal}^{1/3} = 560/30$, hence $\not{D}^2 = \Omega_{\mathfrak{so}(5)} + \frac{49}{20}$. In this way we overlap the general formula of the Casimir operator on a nearly parallel G_2 -manifold, described by I. Agricola and Th. Friedrich in [7, p. 200]; $\Omega = \not{D}^2 - \frac{49}{144} \tau_0^2$ and for $\tau_0 = 6/\sqrt{5}$ it yields the desired result. Using this relation, one concludes that $(\lambda_1^{1/3})^2 \geq \frac{49}{20}$, with the equality appearing if and only if the φ coincides with the spinor field φ_0 . The equality case has been already indicated in Section 3; φ_0 is an eigenspinor of \not{D} with eigenvalue $-\frac{1}{2}\gamma$, hence $(\lambda_1^{1/3})^2 = \frac{1}{4}\gamma^2$ and for $\gamma = -7/\sqrt{5}$, our claim follows. The same states Proposition 6.4 which we shall describe in the final section; the two estimates ought to coincide $\beta_{\text{tw}}(\gamma) = \beta_{\text{univ}}(\gamma) = \frac{7}{54} \text{Scal}^g = \frac{7 \cdot 189}{54 \cdot 10} = \frac{49}{20}$.

Remarks: B. Alexandrov and U. Semmelmann proved in [12, Lem. 7.1] that on a 7-dimensional homogeneous naturally reductive nearly parallel G_2 -manifold $(M = G/K, g, \omega)$, the characteristic connection ∇^c coincides with the canonical connection ∇^1 . Moreover, they showed that if $T^1 = -\frac{7}{6}\omega$ holds for a stable 3-form ω and some $\tau_0 = \text{constant} \neq 0$, then it must be $d\omega = \tau_0 * \omega$ and $\text{Scal}^g = \frac{63}{20\xi^2}$, where $M = G/K$ becomes standard up to the factor ξ^2 , i.e. we assume that g is the normal metric $-\xi^2 B_{\text{SO}_5}|_{\mathfrak{m} \times \mathfrak{m}}$. For B^7 we proved that $\tau_0 = 6/\sqrt{5}$. On the other hand, the used scalar product (\cdot, \cdot) is a multiple of $-B_{\text{SO}_5}$, in particular $-B_{\text{SO}_5} = 6(\cdot, \cdot)$ and since our normal metric $\langle \cdot, \cdot \rangle$ is given by the restriction $(\cdot, \cdot)|_{\mathfrak{m}}$, it follows that $\xi^2 = 1/6$. Thus $\text{Scal}^g = 189/10$, which shows how our computations agree with [12, Lem. 7.1].

5. GEOMETRIC CONSTRAINTS

5.1. Einstein and ∇^c -Einstein structures. We shall now describe the geometric constraints that imposes the existence of a ∇^c -parallel KsT (with respect to $\nabla^s = \nabla^g + 2sT$). We use the same notation with Section 3, i.e. we assume that the triple (M^n, g, T) is endowed with the characteristic connection $\nabla^c = \nabla^g + \frac{1}{2}T$, such that $\nabla^c T = 0$. Let us recall that (M^n, g, T) is said to be ∇^c -Einstein with parallel skew-torsion, if it satisfies the equation $\text{Ric}^c = \frac{\text{Scal}^c}{n}g$ (and $\nabla^c T = 0$)[5]. A special case is when $\text{Ric}^c = 0$ identically; then (M^n, g, T) is called Ric^c -flat. For convenience, we shall henceforth speak for a *strict* ∇^c -Einstein manifold if (M^n, g, T) is ∇^c -Einstein but not Ric^c -flat, i.e. $\text{Ric}^c = \frac{\text{Scal}^c}{n}g \neq 0$.

Proposition 5.1. *Assume that $\nabla^c T = 0$ and that (M^n, g, T) is complete and admits a ∇^c -parallel spinor $0 \neq \varphi \in \Sigma_\gamma$ ($\mathbb{R} \ni \gamma \neq 0$) lying in the kernel $\text{Ker}(P^s)$ for some $s \neq 1/4$. Then, for any $s \in \mathbb{R}$ the following hold*

$$\begin{aligned} \text{Ric}^s(X) \cdot \varphi &= \frac{6\gamma^2}{n^2} \left[\frac{6(n-1)(1-4s)^2 + 96s(1-4s) + 16s(3-4s)(n-3)}{16} \right] X \cdot \varphi, \\ \text{Scal}^s \varphi &= \frac{6\gamma^2}{n} \left[\frac{6(n-1)(1-4s)^2 + 96s(1-4s) + 16s(3-4s)(n-3)}{16} \right] \varphi. \end{aligned}$$

In particular,

- (a) (M^n, g) is a compact Einstein manifold with constant positive scalar curvature $\text{Scal}^g = \frac{9(n-1)\gamma^2}{4n}$.
- (b) For any $n > 3$, (M^n, g, T) is a strict ∇^c -Einstein manifold with parallel torsion and constant scalar curvature $\text{Scal}^c = \frac{3(n-3)\gamma^2}{n}$. For $n = 3$, (M^3, g, T) is Ric^c -flat.
- (c) (M^n, g, T) is ∇^s -Einstein (with non-parallel torsion) for any $s \in \mathbb{R} \setminus \{0, 1/4\}$ i.e. $\text{Ric}^s = \frac{\text{Scal}^s}{n}g$.

Before proceed with a proof of Proposition 5.1, let us remark that due to Theorem 3.7, one can replace the assumptions $\nabla^c \varphi = 0$ and $\varphi \in \text{Ker}(P^s|_{\Sigma_\gamma})$ for some $s \neq 1/4$, with $\nabla^c \varphi = 0$ and $\varphi \in \mathcal{K}^s(M, g)_\zeta$ for some $s \neq 0, 1/4$, where the Killing number is defined by $\zeta := 3(1-4s)\gamma/4n$ for some $0 \neq \gamma \in \text{Spec}(T)$, or $\nabla^c \varphi = 0$ and $\varphi \in \mathcal{K}(M, g)_\kappa$ with Killing number $\kappa = 3\gamma/4n$. Thus, since φ is necessarily a Riemannian Killing spinor with real Killing spinor $3\gamma/4n$, we immediately conclude that (see for example [15, p. 30]):

$$(5.1) \quad \text{Ric}^g(X) \cdot \varphi = 4\kappa^2(n-1)X \cdot \varphi = \frac{9(n-1)\gamma^2}{4n^2}X \cdot \varphi.$$

In particular, (M^n, g) must be Einstein with positive scalar curvature given by $\text{Scal}^g = 4\kappa^2n(n-1) = \frac{9(n-1)\gamma^2}{4n}$. After that, Myers's theorem ensures that M is compact (here we also use that M is complete).

Next we present a different proof of the fact that g is Einstein, without using arguments of the type that such a spinor must be a real Killing spinor. Our computations take place in the spinor bundle and it is very useful to start with a proof of assertion (b) i.e. we provide first the existence of a ∇^c -Einstein structure (and its explicit form), and then we use this fact to describe the original Einstein condition. Notice that this approach differs from the way that ∇^c -Einstein structures have been traditionally examined in dimensions 6 and 7, see [31] and compare with our Examples 5.2 and 5.3 below. Observe also that one is not allowed to apply Corollary 2.3 to compute the Ricci tensor Ric^c , since φ is a Killing spinor with torsion only for $s \neq 0, 1/4$ (recall that we do not view ∇^s -parallel spinors as KsT), and this asserts to our approach a much more special character. Finally, we mention that the ∇^c -Einstein condition plays also a crucial role for the proof of the last claim, since it encodes the action $(X \lrcorner \sigma_T) \cdot \varphi = \frac{3\gamma^2(n-3)}{n^2} X \cdot \varphi$ (recall that φ is always parallel with respect to $\nabla^c = \nabla^g + \frac{1}{2}T$).

Proof. (b) According to Theorem 3.1, the Ricci tensor Ric^c is given by $\text{Ric}^c(X) \cdot \varphi = \frac{1}{2}(X \lrcorner dT) \cdot \varphi = (X \lrcorner \sigma_T) \cdot \varphi$. On the other hand, for any vector field X it holds that [3, p. 325]

$$-2(X \lrcorner \sigma_T) = \frac{1}{2}(T^2 \cdot X - X \cdot T^2) = (X \lrcorner T) \cdot T - T \cdot (X \lrcorner T).$$

To give a few hints for this very useful rule, let us write first $-X \lrcorner \sigma_T = \frac{1}{2}(X \cdot \sigma_T - \sigma_T \cdot X)$. Replacing in the right-hand side the 4-form σ_T by $\sigma_T = \frac{1}{2}(\|T\|^2 - T^2)$, the first relation becomes obvious. Notice now that $X \cdot T + T \cdot X = -2(X \lrcorner T)$. Then, combining the resulting formulae of the Clifford multiplication with T , once from the left and once from the right, we see that $(T^2 \cdot X - X \cdot T^2) = 2((X \lrcorner T) \cdot T - T \cdot (X \lrcorner T))$. Hence, the action of the Ricci endomorphism $\text{Ric}^c(X)$ on ∇^c -parallel spinors reads

$$\text{Ric}^c(X) \cdot \varphi = -\frac{1}{2}[(X \lrcorner T) \cdot T - T \cdot (X \lrcorner T)] \cdot \varphi. \quad (\clubsuit)$$

Now, our assumption tell us that $\varphi \in \text{Ker}(P^s|_{\Sigma_\gamma})$ for some $s \neq 1/4$ and $\gamma \neq 0$; thus, (3.2) needs to hold and this is the key information that the twistor equation carries (of course, due to Theorem 3.7, the same holds if we start with a ∇^c -parallel real Killing spinor φ with $\kappa = 3\gamma/4n$, see also Proposition 3.2). Based on this formula and using $T \cdot \varphi = \gamma\varphi$, one easily computes

$$\begin{aligned} [(X \lrcorner T) \cdot T - T \cdot (X \lrcorner T)] \cdot \varphi &= \gamma(X \lrcorner T) \cdot \varphi + \frac{3\gamma}{n} T \cdot X \cdot \varphi \\ &= -\frac{3\gamma^2}{n} X \cdot \varphi + \frac{3\gamma}{n} (-X \cdot T \cdot \varphi - 2(X \lrcorner T) \cdot \varphi) \\ &= -\frac{3\gamma^2}{n} X \cdot \varphi - \frac{3\gamma^2}{n} X \cdot \varphi - \frac{6\gamma}{n} (X \lrcorner T) \cdot \varphi \\ &= -\frac{6\gamma^2}{n} X \cdot \varphi + \frac{18\gamma^2}{n^2} X \cdot \varphi \\ &= \frac{6\gamma^2}{n} \left[\frac{3}{n} - 1 \right] X \cdot \varphi. \end{aligned}$$

Thus, any $X \in \Gamma(TM)$ satisfies the equation

$$(5.2) \quad \text{Ric}^c(X) \cdot \varphi = \frac{3(n-3)\gamma^2}{n^2} X \cdot \varphi.$$

Now, φ cannot have zeros, since for example by Theorem 3.7 it is also a non-trivial KsT with respect to the family ∇^s and so parallel with respect to the connection $\nabla := \nabla_X^s - \zeta X$, where $\zeta = \frac{3(1-4s)\gamma}{4n} \neq 0$ (and the same occurs for instance since φ is ∇^c -parallel). It follows that for $n > 3$, the triple (M^n, g, T) is a strict ∇^c -Einstein manifold with constant positive scalar curvature $\text{Scal}^c = \frac{3(n-3)\gamma^2}{n}$. For $n = 3$ we get $\text{Ric}^c(X) \cdot \varphi = 0$ for any X , so (M^3, g, T) is Ric^c -flat.

Alternative proof of (a) In the proof of [3, Thm. A.2] (see page 325), for $s = \frac{(n-1)}{4(n-3)}$ and for the family $\nabla_X^s \varphi = \nabla^c + \lambda(X \lrcorner T) \cdot \varphi$ with $\lambda := \frac{1}{2(n-3)}$, the following formula was presented for the curvature tensor

$R^s : \Lambda^2(TM) \rightarrow \text{End}(\Sigma)$ associated to ∇^s :

$$\begin{aligned} \sum_i e_i \cdot R^s(X, e_i)\varphi &= \sum_i e_i \cdot R^c(X, e_i)\varphi - 6\lambda^2(X \lrcorner \sigma_T) \cdot \varphi \\ &\quad - (2\lambda^2 + \lambda) \sum_i T(X, e_i) \cdot (e_i \lrcorner T) \cdot \varphi, \end{aligned}$$

where $\{e_1, \dots, e_n\}$ is an orthonormal frame of M . Exactly the same formula holds for $\lambda = \frac{4s-1}{4}$, i.e. our family ∇^s and for any $s \in \mathbb{R}$. In particular, for $s = \frac{(n-1)}{4(n-3)}$ the quantity $\frac{4s-1}{4}$ is nothing than the fixed $\lambda := \frac{1}{2(n-3)}$. Therefore, in the previous formula one can replace λ by $\frac{4s-1}{4}$ and let s running in \mathbb{R} . Then, for $s = 0$ we get a useful expression between the curvature tensors R^c and R^g associated to the characteristic connection and the Riemannian connection, respectively:

$$(5.3) \quad \sum_i e_i \cdot R^g(X, e_i)\varphi = \sum_i e_i \cdot R^c(X, e_i)\varphi - \frac{6}{16}(X \lrcorner \sigma_T) \cdot \varphi + \frac{1}{8} \sum_i T(X, e_i) \cdot (e_i \lrcorner T) \cdot \varphi.$$

This formula holds for any $\varphi \in \Gamma(\Sigma)$ and $X \in \Gamma(TM)$ and plays a crucial role in what follows.

So, let us emphasize on our case. By assumption $\varphi \in \Sigma_\gamma$ is ∇^c -parallel, hence the first term in the right hand side vanishes, i.e. $\sum_i e_i \cdot R^c(X, e_i)\varphi \equiv 0$ identically. Moreover, we know from (b) that (M^n, g, T) is ∇^c -Einstein with respect to the same metric g , and this determines the second term: $(X \lrcorner \sigma_T) \cdot \varphi = \text{Ric}^c(X) \cdot \varphi = -\frac{3(3-n)\gamma^2}{n^2} X \cdot \varphi$. Finally, for the computation of the third term we take advantage of the fact that $\varphi \in \text{Ker}(P^s|_{\Sigma_\gamma})$ for some $s \neq 1/4$ (which is equivalent to (3.2)). Applying successively this relation, yields

$$\begin{aligned} \sum_i T(X, e_i) \cdot (e_i \lrcorner T) \cdot \varphi &= -\frac{3\gamma}{n} \sum_i T(X, e_i) \cdot e_i \cdot \varphi \\ &= \frac{6\gamma}{n} (X \lrcorner T) \cdot \varphi = -\frac{18\gamma^2}{n^2} X \cdot \varphi. \end{aligned}$$

Thus we obtain altogether:

$$\sum_i e_i \cdot R^g(X, e_i)\varphi = -\frac{9(n-1)\gamma^2}{8n^2} X \cdot \varphi,$$

for any $X \in \Gamma(TM)$. Observing that $\sum_i e_i \cdot R^g(X, e_i)\varphi = -\frac{1}{2} \text{Ric}^g(X) \cdot \varphi$, see [15, p. 15], one can easily finish the proof of part (a).

(c) Now, the last part and the stated formulas for $\text{Ric}^s, \text{Scal}^s$ is an immediate consequence of Corollary 2.3 in combination with Proposition 3.2 and relation $(X \lrcorner \sigma_T) \cdot \varphi = \frac{3\gamma^2(n-3)}{n^2} X \cdot \varphi$, which still makes sense. We mention once more that we apply Corollary 2.3 for $s \neq 0, 1/4$. However, the stated formulas of $\text{Ric}^s, \text{Scal}^s$ produce the right results for any $s \in \mathbb{R}$ (even for $s = 0, 1/4$). This completes the proof. \square

Example 5.2. (see also [31, Prop. 10.4]) Consider a nearly Kähler manifold (M^6, g, J) . Recall that there exist two ∇^c -parallel spinors φ^\pm with $\gamma = \pm 2\|T\|$ which are both TsT for some $s \neq 1/4$. Hence, due to Proposition 5.1 we conclude that

$$\text{Ric}^s(X) \cdot \varphi^\pm = \frac{(5-16s^2)}{4} \|T\|^2 X \cdot \varphi^\pm = \frac{(5-16s^2)}{2} \tau_0 X \cdot \varphi^\pm, \quad \forall s \in \mathbb{R},$$

in particular

$$\begin{aligned} \text{Ric}^c(X) \cdot \varphi^\pm &= \frac{3(n-3)\gamma^2}{n^2} X \cdot \varphi^\pm \Rightarrow \text{Ric}^c(X) \cdot \varphi^\pm = \|T\|^2 X \cdot \varphi^\pm = 2\tau_0 X \cdot \varphi^\pm, \\ \text{Ric}^g(X) \cdot \varphi^\pm &= \frac{9(n-1)\gamma^2}{4n^2} X \cdot \varphi^\pm \Rightarrow \text{Ric}^g(X) \cdot \varphi^\pm = \frac{5}{4} \|T\|^2 \cdot \varphi^\pm = \frac{5}{2} \tau_0 X \cdot \varphi^\pm. \end{aligned}$$

On the other hand, relation (4.1) is still available for a straightforward computation of the Ricci tensor Ric^c . Indeed, a direct computation shows that

$$\begin{aligned} \left[(X \lrcorner T) \cdot T - T \cdot (X \lrcorner T) \right] \cdot \varphi^\pm &= \pm 2 \|T\| (X \lrcorner T) \cdot \varphi^\pm \pm \|T\| T \cdot X \cdot \varphi^\pm \\ &= -2 \|T\|^2 X \cdot \varphi^\pm \pm \|T\| \left(-2 (X \lrcorner T) \cdot \varphi^\pm - X \cdot T \cdot \varphi^\pm \right) \\ &= -2 \|T\|^2 X \cdot \varphi^\pm \pm \|T\| \left(\pm 2 \|T\| X \cdot \varphi^\pm \mp 2 \|T\| X \cdot \varphi^\pm \right) \\ &= -2 \|T\|^2 X \cdot \varphi^\pm, \end{aligned}$$

and the result now is an immediate consequence of (♣). In this spinorial way and for $s = 0, 1/4$, we overlap the results of [31, Prop. 10.4]. Mention however that our method is different, i.e. we are *not* based on the Einstein property of g and the relation $\text{Ric}^c = \text{Ric}^g - \frac{1}{4}S$. Finally, the type $\text{Ric}^s = \text{Ric}^g - 4s^2S$ produces the same results for all $s \in \mathbb{R}$, since $S := 2\tau_0 \text{Id}$, see [31, Prop. 10.4].

Example 5.3. (see also [31, Thm. 5.1, Ex. 5.2]) Consider a nearly parallel G_2 -manifold (M^7, g, ω) . Recall that there is a unique ∇^c -parallel spinor field φ_0 with $\gamma = -\sqrt{7}\|T\|$. In a similar way with Example 5.2 and due to Proposition 5.1, one gets that

$$\text{Ric}^s(X) \cdot \varphi_0 = \frac{6(9 - 16s^2)}{28} \|T\|^2 X \cdot \varphi_0 = \frac{(9 - 16s^2)}{24} \tau_0^2 X \cdot \varphi_0, \quad \forall s \in \mathbb{R},$$

in particular

$$\text{Ric}^c(X) \cdot \varphi_0 = \frac{12}{7} \|T\|^2 X \cdot \varphi_0 = \frac{\tau_0^2}{3} X \cdot \varphi_0, \quad \text{Ric}^g(X) \cdot \varphi_0 = \frac{27}{14} \|T\|^2 X \cdot \varphi_0 = \frac{3\tau_0^2}{8} X \cdot \varphi_0.$$

In a line with nearly Kähler manifolds in dimension 6, relation (4.3) gives us the ability to compute Ric^c in a direct way. Indeed, due to (4.2) and (4.3), a little computation shows that

$$\left[(X \lrcorner T) \cdot T - T \cdot (X \lrcorner T) \right] \cdot \varphi_0 = -\frac{24}{7} \|T\|^2 X \cdot \varphi_0,$$

and the result follows by (♣). Notice finally that one can reproduce all these results, using the formula $\text{Ric}^s = \text{Ric}^g - 4s^2S$, see [31, Ex. 5.2] for the tensor $S := \frac{\tau_0^2}{6} \text{Id}$.

5.2. The 3-dimensional case. In [6, Ex. 7.2] it is shortly explained that the *unique* 3-dimensional compact manifold carrying parallel spinors with respect to a metric connection with non-trivial skew-torsion, is a space form, namely the round 3-sphere $S^3 = \text{Spin}_4 / \text{Spin}_3$ endowed with the canonical spin structure $(P, \Lambda) := (\text{Spin}_3 \times \text{Spin}_3, \text{Id}_{\text{Spin}_3} \times \lambda)$ where $\lambda : \text{Spin}_3 \rightarrow \text{SO}_3$ is the double covering, the canonical metric $g := g_{\text{can}}$ of constant sectional curvature 1 and finally the volume form $T = \text{Vol}_{S^3}$. But let us explain how this can fit with our results and what new we can say. The spinor bundle $\Sigma := P \times_{\text{SU}_2} \Delta_3 \rightarrow S^3$ is trivialized through either the $-\frac{1}{2}$ - or $\frac{1}{2}$ -Killing spinors and T acts as the identity operator (and as a scalar operator in the case that $T = f \text{Vol}_{S^3}$, for some constant f). By [6] we know that the ϵ -Killing spinors are parallel with respect to the metric connections induced by the Killing spinor equation: $\nabla_X^\epsilon \varphi = \nabla_X^g \varphi + \epsilon (X \lrcorner T) \cdot \varphi = \nabla_X^g \varphi - \epsilon X \cdot T \cdot \varphi = \nabla_X^g \varphi - \epsilon X \cdot \varphi = 0$, where $\epsilon \in \{\pm \frac{1}{2}\}$. For convenience, we set $T^\epsilon := 2T$ and rewrite

$$\nabla_X^\epsilon \varphi = \nabla_X^g \varphi + \frac{\epsilon}{2} (X \lrcorner T^\epsilon) \cdot \varphi = \nabla_X^g \varphi \pm \frac{1}{4} (X \lrcorner T^\epsilon) \cdot \varphi = \nabla_X^g \varphi \mp \frac{1}{4} X \cdot T^\epsilon \cdot \varphi = \nabla_X^g \varphi \mp \frac{1}{2} X \cdot \varphi.$$

Now it is obvious that $\nabla^\epsilon \equiv \nabla^{\pm 1/2}$ are metric connections with skew-torsion

$$\pm T^\epsilon = \pm 2T = \pm 2(e_1 \wedge e_2 \wedge e_3),$$

such that $\nabla^\epsilon T^\epsilon = 0$ (since S^3 is orientable and T is the volume form). On $T S^3$ one can write $\nabla^\epsilon = \nabla^g \pm \frac{1}{2} T^\epsilon$. Notice that since the sphere S^3 is diffeomorphic to the compact Lie group $\text{Spin}_3 \cong \text{SU}_2 \cong \text{Sp}_1$ a characteristic (or canonical) connection is not unique [44, 43, 9]. Endowed with a bi-invariant metric and one of the \pm -canonical connections of Cartan-Schouten is flat and hence also Ric^\pm -flat, see for instance [6, Ex. 7.1]. ∇^ϵ is also flat, in particular S^3 is simply connected and the flatness of ∇^ϵ implies the triviality of the associated spinor bundle Σ [13, Lem. 1].

Lemma 5.4. $T^\epsilon(X, Y) \cdot \varphi = -(X \cdot Y - Y \cdot X) \cdot \varphi$, for any vector field X, Y and spinor field φ .

Proof. 1st way. In [13] appears the following expression (for $\epsilon = 1/2$ and for a general sphere S^n)

$$\nabla_X^\epsilon \nabla_Y^\epsilon \varphi = (\nabla_X^g - \frac{1}{2}X)(\nabla_Y^g - \frac{1}{2}Y)\varphi = \nabla_X^g \nabla_Y^g \varphi - \frac{1}{2}Y \cdot \nabla_X^g \varphi - \frac{1}{2}X \cdot \nabla_Y^g \varphi + \frac{1}{4}X \cdot Y \cdot \varphi,$$

where locally for the Riemannian connection one can assume that $(\nabla^g X)(p) = (\nabla^g Y)(p) = 0$, for some vector fields $X, Y \in \Gamma(TS^3)$ and $p \in S^3$, see also [15, p. 23]. However, the same time we can write

$$\begin{aligned} \nabla_X^\epsilon \nabla_Y^\epsilon \varphi &= \nabla_X^\epsilon (\nabla_Y^g \varphi - \frac{1}{2}Y \cdot \varphi) = \nabla_X^\epsilon \nabla_Y^g \varphi - \frac{1}{2} \nabla_X^\epsilon (Y \cdot \varphi) \\ &= \nabla_X^g \nabla_Y^g \varphi - \frac{1}{2}X \cdot (\nabla_Y^g \varphi) - \frac{1}{2}(\nabla_X^\epsilon Y) \cdot \varphi - \frac{1}{2}Y \cdot \nabla_X^g \varphi + \frac{1}{4}Y \cdot X \cdot \varphi. \quad (\star) \end{aligned}$$

A comparison now with the previous relation shows that $(\nabla_X^\epsilon Y) \cdot \varphi = -\frac{1}{2}(X \cdot Y - Y \cdot X) \cdot \varphi$. Hence, after replacing $\nabla_X^\epsilon Y = \nabla_X^g Y + \frac{1}{2}T^\epsilon(X, Y)$, we get our assertion:

$$T^\epsilon(X, Y) \cdot \varphi = -(X \cdot Y - Y \cdot X) \cdot \varphi.$$

Similarly for the case of $\epsilon = -1/2$, i.e. the connection $\nabla_X^{-1/2} \varphi = \nabla_X^g \varphi + \frac{1}{2}X \cdot \varphi$ with torsion $-T^\epsilon$.

2nd way. (We again explain the case $\epsilon = 1/2$). Notice that one can avoid the assumption $(\nabla^g X)(p) = (\nabla^g Y)(p) = 0$, since (\star) itself is independent of this condition. In the same way, we write

$$\nabla_Y^\epsilon \nabla_X^\epsilon \varphi = \nabla_Y^g \nabla_X^g \varphi - \frac{1}{2}Y \cdot (\nabla_X^g \varphi) - \frac{1}{2}(\nabla_Y^\epsilon X) \cdot \varphi - \frac{1}{2}X \cdot \nabla_Y^g \varphi + \frac{1}{4}X \cdot Y \cdot \varphi$$

and as a consequence of the definition $R^\epsilon(X, Y)\varphi = \nabla_X^\epsilon \nabla_Y^\epsilon \varphi - \nabla_Y^\epsilon \nabla_X^\epsilon \varphi - \nabla_{[X, Y]}^\epsilon \varphi$, we see that (we give the expression for both $\epsilon \in \{\pm 1/2\}$)

$$\begin{aligned} R^\epsilon(X, Y)\varphi &= R^g(X, Y)\varphi \mp \frac{1}{2} \left[\nabla_X^\epsilon Y - \nabla_Y^\epsilon X - [X, Y] \right] \cdot \varphi + \frac{1}{4}(Y \cdot X - X \cdot Y) \cdot \varphi \\ (5.4) \quad &= R^g(X, Y)\varphi - \frac{1}{2}T^\epsilon(X, Y)\varphi + \frac{1}{4}(Y \cdot X - X \cdot Y) \cdot \varphi. \end{aligned}$$

Here, $\nabla_X^\epsilon Y - \nabla_Y^\epsilon X - [X, Y]$ equals to $\pm T^\epsilon(X, Y)$, depending on $\epsilon \in \{\pm 1/2\}$. Now, based on the fact that S^3 has constant sectional curvature 1 one computes $R^g(X, Y)\varphi = \frac{1}{4}(Y \cdot X - X \cdot Y) \cdot \varphi$, see [13] (or [15, Thm. 8, p. 30]). Combining this with $R^\epsilon \equiv 0$, relation (5.4) gives rise to the desired result. \square

Let us identify the tangent space $T_o S^3$ with $\mathfrak{g} := \mathfrak{su}(3) = \mathfrak{spin}(3) = \mathfrak{so}(3)$ and denote by $\{e_1, e_2, e_3\}$ the left-invariant vector fields associated to a basis of \mathfrak{g} . Then, $T^\epsilon(e_i, e_j) \cdot \varphi = -(e_i \cdot e_j - e_j \cdot e_i) \cdot \varphi = -[e_i, e_j] \cdot \varphi$ holds for any $\varphi \in \Gamma(\Sigma)$. Because φ can be written as a linear combination of Killing spinors, φ has no-zeros and thus $T^\epsilon(e_i, e_j) = -[e_i, e_j]$. Hence, this spinorial approach allows us to deduce that the torsion $\pm T^\epsilon$ coincides with the torsion of the ± 1 -Cartan-Schouten connections, as it should be due to the uniqueness of $\nabla^{\pm 1}$.

Relation (5.4) has another remarkable application; it implies that (S^3, g) is Einstein, without using arguments of the type that S^3 is a space form in dimension 3, neither a manifold carrying Killing spinors, nor an isotropy irreducible homogeneous space. In fact, we do not even use the flatness of ∇^ϵ , but only the fact that given a trivialization $\{\varphi_j : 1 \leq j \leq 2^{\lfloor \frac{3}{2} \rfloor}\}$ of Σ by ϵ -Killing spinors, then the relation $\nabla^\epsilon \varphi_j = 0$ needs to hold $\forall j$ (in a similar way with nearly Kähler and nearly parallel G_2 -manifolds). Indeed, since $T^\epsilon \cdot \varphi_j = 2T \cdot \varphi_j = 2\varphi_j$, applying Clifford multiplication on (5.4) with respect to the orthonormal frame $\{e_1, e_2, e_3\}$ and finally adding, yields

$$\begin{aligned} \sum_i e_i \cdot R^\epsilon(X, e_i)\varphi_j &= \sum_i e_i \cdot R^g(X, e_i)\varphi_j - \frac{1}{2} \sum_i e_i \cdot T^\epsilon(X, e_i) \cdot \varphi_j + \frac{1}{4} \sum_i e_i \cdot (e_i \cdot X - X \cdot e_i) \cdot \varphi_j \\ &= -\frac{1}{2} \text{Ric}^g(X) \cdot \varphi_j - (X \lrcorner T^\epsilon) \cdot \varphi_j - X \cdot \varphi_j = -\frac{1}{2} \text{Ric}^g(X) \cdot \varphi_j + X \cdot T^\epsilon \cdot \varphi_j - X \cdot \varphi_j \\ &= -\frac{1}{2} \text{Ric}^g(X) \cdot \varphi_j + 2X \cdot \varphi_j - X \cdot \varphi_j. \end{aligned}$$

Because $\nabla^{\pm 1/2} \varphi_j = 0$, the left-hand side vanishes and the resulting formulae $\text{Ric}^g(X) \cdot \varphi_j = 2X \cdot \varphi_j$ shows that (S^3, g_{can}) is Einstein with Einstein constant $2 = \text{Scal}^g/n = 6/3$, see also [25].

Now, because ∇^ϵ are flat, (S^3, g_{can}) is automatically Ric^ϵ -flat. This coincides with the statement of Proposition 5.1 (here we only allow $\epsilon = 1/2$), namely

$$\begin{aligned}\text{Ric}^g(X) \cdot \varphi_j &= \frac{9(n-1)\gamma^2}{4n^2} X \cdot \varphi_j \Rightarrow \text{Ric}^g(X) \cdot \varphi_j = 2X \cdot \varphi_j, \\ \text{Ric}^\epsilon(X) \cdot \varphi_j &= \frac{3(n-3)\gamma^2}{n^2} X \cdot \varphi_j \Rightarrow \text{Ric}^\epsilon(X) \cdot \varphi_j = 0.\end{aligned}$$

More generally, $\text{Ric}^s(X) \cdot \varphi_j = 2(1 - 16s^2)X \cdot \varphi_j$, i.e.

$$\text{Ric}^s = 2(1 - 16s^2)\text{Id}, \quad \forall s \in \mathbb{R}.$$

Of course, and with the aim to apply Proposition 5.1, one has to consider first the family

$$\nabla_X^{\epsilon,s} \varphi := \nabla_X^g \varphi + s(X \lrcorner T^\epsilon) \cdot \varphi.$$

For $s = \pm 1/4$ it induces the flat connections $\nabla^{\epsilon, \pm 1/4} = \nabla^\epsilon = \nabla^{\pm 1/2}$ and for $s = 0$ it coincides with the spinorial Riemannian connection. Because the trivialization $\{\varphi_j : 1 \leq j \leq 2^{\lfloor \frac{3}{2} \rfloor}\}$ of Σ consists of ϵ -Killing spinors which are ∇^ϵ -parallel, Theorem 3.7 states that

Theorem 5.5. *There is a one-to-one correspondence between ϵ -Killing spinors on $(S^3, g_{\text{can}}, T^\epsilon)$ and Killing spinors with torsion with respect to the family $\nabla^{\epsilon,s}$ for any $s \neq 0, 1/4$, with Killing number $\zeta = \frac{1-4s}{2}$, i.e. $\nabla_X^{\epsilon,s} \varphi_j = \frac{1-4s}{2} X \cdot \varphi_j$, $\forall X \in \Gamma(TS^3)$. In particular, a 3-dimensional compact spin manifold (M^3, g, T) satisfying the assumptions of Proposition 5.1, is isometric to $(S^3, g_{\text{can}}, T^\epsilon)$.*

Proof. Let us shortly present a direct proof. If $\nabla_X^g \varphi = \frac{1}{2} X \cdot \varphi$ for any $X \in \Gamma(TS^3)$, then

$$\begin{aligned}\nabla_X^{\epsilon,s} \varphi &= \frac{1}{2} X \cdot \varphi + s(X \lrcorner T^\epsilon) \cdot \varphi = \frac{1}{2} X \cdot \varphi + 2s(X \lrcorner T) \cdot \varphi \\ &= \frac{1}{2} X \cdot \varphi - 2sX \cdot T \cdot \varphi = \frac{1-4s}{2} X \cdot \varphi.\end{aligned}$$

Conversely, if $\varphi \in \mathcal{K}^s(S^3, g)_\zeta$ with $\zeta = \frac{1-4s}{2}$ for some $s \neq 0, 1/4$, then $\nabla_X^{\epsilon,s} \varphi = \zeta X \cdot \varphi$ and thus

$$\nabla_X^g \varphi = \zeta X \cdot \varphi - s(X \lrcorner T^\epsilon) \cdot \varphi = \frac{1-4s}{2} X \cdot \varphi + 2sX \cdot T \cdot \varphi = \frac{1}{2} X \cdot \varphi.$$

□

5.3. A partial classification. We deduce that on a triple (M^n, g, T) with $\nabla^c T = 0$, the existence of a spinor field φ satisfying simultaneously the equations

$$(5.5) \quad \nabla_X^c \varphi = 0, \quad \nabla_X^s \varphi = \zeta X \cdot \varphi, \quad \forall X \in \Gamma(TM),$$

for some real numbers $s \neq 0, 1/4$, $\zeta \neq 0$, where $\nabla^s = \nabla^g + 2sT$, imposes much harder geometric restrictions than the original Killing spinor equation, namely:

	TYPE OF KILLING SPINORS	GEOMETRIC CONCLUSIONS
(α)	Killing spinors with Killing number $\kappa \in \mathbb{R} \setminus \{0\}$	• $\text{Ric}^g = 4\kappa^2(n-1)g$, $\text{Scal}^g = 4\kappa^2 n(n-1)$
(β)	∇^c -parallel KsT w.r.t. $\nabla^s = \nabla^g + 2sT$ with Killing number $\zeta = \frac{3(1-4s)\gamma}{n} \neq 0$ for some $\mathbb{R} \ni \gamma \neq 0$, $\mathbb{R} \ni s \neq 0, 1/4$	• φ is a real Killing spinor: $T \cdot \varphi = \gamma \cdot \varphi \neq 0$ • $\text{Ric}^s = \frac{\text{Scal}^s}{n} g \quad \forall s \in \mathbb{R}$, in particular : – $\text{Ric}^g = \frac{9(n-1)\gamma^2}{4n^2} g$, $\text{Scal}^g = \frac{9(n-1)\gamma^2}{4n}$ – $\text{Ric}^c = \frac{3(n-3)\gamma^2}{n^2} g$, $\text{Scal}^c = \frac{3(n-3)\gamma^2}{n}$

One has to stress that this is not the case in general; there exist Killing spinors with torsion (KsT) which are not real Killing spinors, and thus manifolds which are not necessarily Einstein can be endowed with them, e.g. the Heisenberg group, see [17, pp. 54–57] and [10].

We conclude that there are several examples of special structures endowed with their characteristic connection which *fail* to carry this special kind of KsT (or TsT). Actually, since such a ∇^c -parallel KsT must be finally a real Killing spinor, we need only to focus on special structures carrying Riemannian Killing spinors. Such structures have been classified in dimensions $4 \leq n \leq 8$ by Th. Friedrich's school in Berlin, see [25], [26], [30], [32], [38], [33] and [34]. According to [33, Thm. 1], any Einstein-Sasakian manifold $(M^{2m+1}, g, \xi, \eta, \phi)$ admits real Killing spinors. In particular, in dimension 5 such manifolds

together with the standard sphere exhaust all possible cases [32]. In dimension 7, the special structure which carries real Killing spinors is necessarily a nearly parallel G_2 -structure, see [34] for the three different types. Finally, for higher odd dimensions $4m+1 \geq 9, 4m+3 \geq 11$ we know by [14] that only spheres, Einstein-Sasakian manifolds and 3-Sasakian manifolds can admit this special kind of spinors, while in even dimensions, beyond the 6-dimensional nearly Kähler manifolds (see [30, 38]), the unique members are the standard spheres. We also refer to [36, p. 143] for a summary of all these results.

Notice now that [5, Lem. 2.23] states that if an almost contact metric structure $(M^{2m+1}, g, \xi, \eta, \phi)$ is ∇^c -Einstein with respect to a characteristic connection, then it must be ∇^c -Ricci flat. So, such a manifold is never strict ∇^c -Einstein. In particular, an Einstein-Sasaki manifold M^{2m+1} cannot be ∇^c -Einstein, see [5, Rem. 2.26], hence compact Einstein-Sasakian spin manifolds $(M^{2m+1}, g, \xi, \eta, \phi)$ in any odd dimension $n = 2m+1 \geq 5$, although manifolds with real Killing spinors, cannot carry a ∇^c -parallel KsT with respect to $\nabla^s = \nabla^g + 2sT$. The same comes true for the Tanno deformation of an Einstein-Sasakian manifold (see below for the Tanno deformation); in the best case there is a specific parameter $t = t_o$ which makes $(M^{2m+1}, g_t, \xi_t, \eta_t, \phi)$, Ric^{∇^t} -flat with respect to the induced characteristic connection $\nabla^t = \nabla^{g_t} + \frac{1}{2}\eta_t \wedge d\eta_t$, see [5, Thm. 2.24]. This shows that the ∇^c -Einstein condition is still very restrictive and our KsT are very special. We remark that in dimension 5, and for general KsT a similar “non-existence” result has been described (using another integrability condition) in [3, Cor. A.2].

After this discussion and due to Proposition 5.1 and Theorems 4.1, 4.2, 5.5, we summarise as follows:

Theorem 5.6. *Let (M^n, g, T) be a compact connected Riemannian spin manifold with $\nabla^c T = 0$, endowed with a spinor field satisfying (5.5) with respect to the same Riemannian metric g . If $n = 3$, then $M^3 \cong S^3$ is isometric to the 3-sphere. If $n = 6$, then M^6 is isometric to a strict nearly Kähler manifold. If $n = 7$, then M^7 is isometric to a nearly parallel G_2 -manifold.*

It is an interesting question the existence of an analogue of Theorem 5.5 for some even dimensional sphere S^{2m} (different that $S^6 = G_2/SU_3$). For S^4 this cannot be the case due to [24, Thm. 1.1].

A different construction. With the aim to avoid confusions, we recall that J. Becker-Bender in her Phd thesis proved that Killing spinors with torsion on Einstein-Sasakian manifolds exist [17, Cor. 2.18]; they appear after deforming the metric associated to the real Killing spinors by applying the Tanno transformation, see also [3, Ex. 5.1, 5.2] and [10, p. 21]. Let us shortly explain the *difference* of this construction with the present work. For an almost contact manifold $(M^{2m+1}, g, \xi, \eta, \phi)$ the Tanno deformation [45] is given by $g_t = tg + (t^2 - t)\eta \otimes \eta$, $\xi_t = \frac{1}{t}\xi$ and $\eta_t = t\eta$ for some $t > 0$ (for details on Sasakian geometry see [45, 33, 34, 18, 31, 2]). If $(M^{2m+1}, g, \xi, \eta, \phi)$ is Sasakian, then $(M^{2m+1}, g_t, \xi_t, \eta_t, \phi)$ is too [45, 35]. Consider the Tanno deformation of an Einstein-Sasakian manifold $(M^{2m+1}, g, \xi, \eta, \phi)$ with $2m+1 \geq 5$. In [17, Thm. 2.22] it was shown that $(M^{2m+1}, g_t, \xi_t, \eta_t, \phi)$ admits Killing spinors with torsion for the parameters $s_t = \frac{m+1}{4(m-1)}(\frac{1}{t} - 1)$, with respect to the connection

$$\nabla_X^{s_t} \varphi = \nabla_X^{g_t} \varphi + s_t (X \lrcorner T^c) \cdot \varphi, \quad T^c := \eta_t \wedge d\eta_t = 2\eta_t \wedge F_t,$$

with Killing numbers $\zeta_{1,t} = \frac{\varepsilon}{2}(1 - 4s_t)$ and $\zeta_{2,t} = (-1)^{m+1}\zeta_{1,t}$, respectively. Here, $\varepsilon = \pm 1$ is the number defined by the equation $e_1 \cdot \phi(e_1) \cdot \dots \cdot e_m \cdot \phi(e_m) \cdot \xi \cdot \varphi = \varepsilon i^{m+1} \varphi$ for a local orthonormal frame $\{e_1, \phi(e_1), \dots, e_m, \phi(e_m), \xi\}$ of M^{2m+1} . If there is no deformation ($t = 1$), then $s_t = 0$ and φ_i coincide with the Riemannian Killing spinors that $(M^{2m+1}, g, \xi, \eta, \phi)$ carries, see [33, Thm. 1]. For $1 - 4s_t = 0$, i.e. the Riemannian metric $g_{t_o} = g_{m+1/2m}$, the spinor fields are ∇^c -parallel, see also [10, p. 21]. However, φ_i are KsT for any Riemannian metric g_t with $t > 0, t \neq \frac{m+1}{2m}$. This shows that the equations given in (5.5) hold with respect to different Riemannian metrics. Maybe it is an interesting but difficult task to describe new non-integrable G -structures carrying Killing spinors with torsion after a compatible deformation of the given G -structure, if any.

6. FURTHER APPLICATIONS

6.1. A converse direction of Proposition 5.1. In the proof of Proposition 5.1 we proved that a ∇^c -parallel twistor spinor with torsion $\varphi \in \text{Ker}(P^s|_{\Sigma_\gamma})$ for some $\gamma \neq 0$ and $s \neq 1/4$, satisfies the equation

$$(6.1) \quad \sum_i T(X, e_i) \cdot (e_i \lrcorner T) \cdot \varphi = -\frac{18\gamma^2}{n^2} X \cdot \varphi, \quad \forall X \in \Gamma(TM).$$

Next we will show that (6.1) is still true if:

Lemma 6.1. *Let (M^n, g, T) ($n > 3$) be a (compact) Riemannian manifold with $\nabla^c T = 0$, carrying a ∇^c -parallel spinor field $0 \neq \varphi \in \Sigma_\gamma$ for some $0 \neq \gamma \in \text{Spec}(T)$, where $\nabla^c = \nabla^g + \frac{1}{2}T$ is the characteristic connection. Assume that M^n is both Einstein and ∇^c -Einstein with respect to g , in particular that the relations (5.1) and (5.2) are satisfied for any $X \in \Gamma(TM)$. Then, any vector field $X \in \Gamma(TM)$ satisfies (6.1), as well.*

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal frame of M^n with respect to g . Under our assumptions the following relations are true for some $\gamma \neq 0$: $\sum_i e_i \cdot R^g(X, e_i)\varphi = -\frac{1}{2}\text{Ric}^g(X) \cdot \varphi = -\frac{9(n-1)\gamma^2}{8n^2}X \cdot \varphi$ and $(X \lrcorner \sigma_T) \cdot \varphi = \text{Ric}^c(X) \cdot \varphi = \frac{3(n-3)\gamma^2}{n^2}X \cdot \varphi$. Then, equation (6.1) it is a simple consequence of (5.3) and the ∇^c -parallelism of φ . \square

Under the assumptions of Lemma 6.1 and for general n , we are not able to show that equation (6.1) implies the twistor equation. However, this is possible for $n = 6$ and a nearly Kähler manifold, or $n = 7$ and a nearly parallel G_2 -manifold, in combination with the relations (4.1) and (4.3), respectively. Notice that although in our text these formulas appear as a consequence of the twistor equation with torsion, both are deeper consequences of the ∇^c -parallelism of the spinors φ^\pm (respectively φ_0) and the character of the spin representation in these dimensions, see for example [4, Lem. 2.2, 2.3]. Further applications of the relations (4.1) and (4.3) for these two kinds of weak holonomy structures, will be shortly described in an appendix.

In [3], a full integrability condition for the existence of Killing spinors with torsion with respect to the family ∇^s for $s = \frac{n-1}{4(n-3)}$ was presented. For convenience, let us recall the statement.

Theorem 6.2. ([3, Thm. A.2, p. 324]) *Let φ be a KsT with respect to ∇^s for $s = \frac{n-1}{4(n-3)}$ with Killing number ζ . Set $\lambda := 1/2(n-3)$. Then*

$$(6.2) \quad \begin{aligned} \text{Ric}^c(X) \cdot \varphi &= -16s\zeta(X \lrcorner T) \cdot \varphi + 4(n-1)\zeta^2 X \cdot \varphi + (1-12\lambda^2)(X \lrcorner \sigma_T) \cdot \varphi \\ &\quad - 2(2\lambda^2 + \lambda) \sum_i T(X, e_i) \cdot (e_i \lrcorner T) \cdot \varphi. \end{aligned}$$

For a Riemannian manifold (M^n, g, T) with $\nabla^c T = 0$ one can use Theorem 3.7 and Proposition 5.1 to prove that if $\varphi \in \mathcal{K}^s(M, g)_\zeta$ is a KsT for $s = \frac{n-1}{4(n-3)} \neq 0, 1/4$ with Killing number $\zeta = 3\gamma(1-4s)/4n$, which is in addition ∇^c -parallel w.r.t. the characteristic connection $\nabla^c = \nabla^g + \frac{1}{2}T$, then (6.2) reduces to an identity, namely the twistor equation with torsion. In fact, for such ∇^c -parallel KsT w.r.t. $\nabla^s = \nabla^g + 2sT$, the optimal integrability conditions are these described by Theorem 5.1.

Corollary 6.3. *Let (M^n, g, T) ($n > 3$) be a compact connected Riemannian manifold with $\nabla^c T = 0$, where $\nabla^c = \nabla^g + \frac{1}{2}T$ is the characteristic connection. Consider a ∇^c -parallel Killing spinor with torsion $\varphi_0 \in \mathcal{K}^s(M^n, g)_\zeta$ for $s = \frac{n-1}{4(n-3)} \neq 0, 1/4$, with Killing number $\zeta = 3\gamma(1-4s)/4n$ for some $0 \neq \gamma \in \text{Spec}(T)$. Then, φ_0 satisfies the condition (6.2) as an identity.*

Proof. Consider a spinor field $\varphi_0 \in \Gamma(\Sigma)$ which is ∇^c -parallel and the same time a Killing spinor with torsion for $s = \frac{n-1}{4(n-3)} \neq 0, 1/4$. By assumption, the Killing number ζ is given by $\zeta = \frac{3(1-4s)\gamma}{4n}$, i.e. $\zeta = -\frac{3\gamma}{2n(n-3)}$ for some T -eigenvalue $\gamma \neq 0$. Automatically, φ is an element in the kernel of the twistor operator P^s for some $s \neq 1/4$, thus according to Proposition 5.1, the relations (5.1) and (5.2) are satisfied for any $X \in \Gamma(TM)$. Moreover, by the proof of the same proposition (or Lemma 6.1, since the condition $\varphi \in \Sigma_\gamma$ follows by the twistor equation) we know that also relation (6.1) needs to hold. Combining this information, is a straightforward computation to show that the integrability condition given by (6.2) reduces to $(X \lrcorner T) \cdot \varphi_0 + \frac{3\gamma}{n}X \cdot \varphi_0 = 0, \forall X \in \Gamma(TM)$. \square

6.2. The relation between the two estimates and the role of Proposition 3.2. For the first eigenvalue of \mathcal{D}^2 restricted on Σ_γ there is a second estimate, the so-called *twistorial estimate* introduced in [3, 17]. This is defined by

$$(6.3) \quad \lambda_1(\mathcal{D}^2|_{\Sigma_\gamma}) \geq \frac{n}{4(n-1)} \text{Scal}_{\min}^g + \frac{n(n-5)}{8(n-3)^2} \|T\|^2 + \frac{n(4-n)}{4(n-3)^2} \gamma^2 := \beta_{\text{tw}}(\gamma),$$

and the equality case appears if and only if φ is a twistor spinor with torsion for $s = (n-1)/4(n-3)$ and $\text{Scal}^g = \text{constant}$. A similar expression with (6.3) holds for the whole spinor bundle. In the presence of a ∇^c -parallel spinor $0 \neq \varphi \in \Sigma_\gamma$, the inequality $\beta_{\text{tw}}(\gamma) \leq \beta_{\text{univ}}(\gamma)$ needs to hold. For $n \leq 8$, this fact implies the inequalities [3, Lem. 4.1]

$$(6.4) \quad 0 \leq 2n\|T\|^2 + (n-9)\gamma^2, \quad \text{Scal}^g \leq \frac{9(n-1)}{2(9-n)}\|T\|^2.$$

The equality case here, takes place if and only if the universal estimate coincides with the twistorial estimate. However, if one of these inequalities holds as an equality, then Proposition 3.2 states that φ must be a real Killing spinor. In fact, one can say more:

Proposition 6.4. *Let (M^n, g, ∇^c) ($3 < n \leq 8$) be a compact connected Riemannian spin manifold with $\nabla^c T = 0$ and positive scalar curvature, carrying a spinor field $\varphi \in \Gamma(\Sigma)$ satisfying the equations given by (3.1) for some $0 \neq \gamma \in \text{Spec}(T)$. If $\beta_{\text{tw}}(\gamma) = \beta_{\text{univ}}(\gamma)$ then φ is a real Killing spinor with respect to g , with Killing number $\kappa = 3\gamma/4n$. Conversely, if φ is a real Killing spinor with $\kappa = 3\gamma/4n$ satisfying (3.1), then $\beta_{\text{tw}}(\gamma) = \beta_{\text{univ}}(\gamma)$ identically.*

Proof. Since $\beta_{\text{tw}}(\gamma) = \beta_{\text{univ}}(\gamma)$ if and only if one of the above inequalities holds as equality, the one direction is obvious due to Proposition 3.2, which is their equality case. In fact, under our assumptions, the scalar curvature Scal^g is constant (see Theorem 3.1) and the universal estimate becomes sharp. Since $\not{D} = D^c - \frac{1}{2}T$, the spinor φ is an eigenspinor of the cubic Dirac operator with eigenvalue $-\frac{1}{2}\gamma$; therefore $\lambda_1 = \frac{1}{4}\gamma^2 = \beta_{\text{univ}}(\gamma)$. Since the two estimates coincide, we conclude that $\beta_{\text{tw}}(\gamma) = \frac{1}{4}\gamma^2$ also, where $\beta_{\text{tw}}(\gamma)$ is given now by the equality case in (6.3) and without the minimal condition in Scal^g . Hence, for this direction one can also present another proof based on Theorem 3.7. According to [3], if the two estimates coincide (and both hold with the equality), then the spinor $\varphi \in \Sigma_\gamma$ must be ∇^c -parallel and also a twistor spinor with torsion for $s = (n-1)/4(n-3)$. Then, by Theorem 3.7, φ is also a Killing spinor with torsion with Killing number $\zeta = \frac{3(1-4s)\gamma}{4n} = -\frac{3\gamma}{2n(n-3)}$ and moreover a real Killing spinor with Killing number $\kappa = 3\gamma/4n$.

We present a proof for the converse direction. If φ is a real Killing spinor on (M^n, g, T) with $\kappa = 3\gamma/4n$, then $\text{Scal}^g = \frac{9(n-1)}{4n}\gamma^2$ is constant; because $\varphi \in \Sigma_\gamma$ is ∇^c -parallel and the scalar curvature satisfies the desired formula, by Proposition 3.2 for $n \leq 8$ we also have $\|T\|^2 = \frac{2(9-n)}{9(n-1)}\text{Scal}^g$. Moreover, Theorem 3.7 tell us that this is also a twistor spinor for some $s \neq 1/4$; thus one may assume without loss of generality that $s = (n-1)/4(n-3) \neq 1/4$. Hence, finally the twistorial estimate must hold as an equality and replacing the previous values, we obtain the expression

$$\beta_{\text{tw}}(\gamma) = \frac{n[9(n-3)^2 + (n-5)(9-n) + 4n(4-n)]}{36(n-1)(n-3)^2} \text{Scal}^g.$$

Equivalent expressions in terms only of $\|T\|^2$ or γ^2 , can be easily deduced. The same time, $\varphi \in \Sigma_\gamma$ is ∇^c -parallel, hence for the universal estimate we get $\beta_{\text{univ}}(\gamma) = \frac{\gamma^2}{4} = \frac{n}{9(n-1)}\text{Scal}^g$. Then, one can easily check that the equation $\beta_{\text{tw}}(\gamma) - \beta_{\text{univ}}(\gamma) = 0$ holds as an identity. \square

Example 6.5. For $n = 7$ and for a nearly parallel G_2 -manifold (M^7, g, ω) we compute $\beta_{\text{tw}}(\gamma) = \frac{7}{54}\text{Scal}^g = \beta_{\text{univ}}(\gamma)$. For $n = 6$ and a nearly Kähler manifold (M^6, g, J) we have $\beta_{\text{tw}}(\gamma) = \frac{2}{15}\text{Scal}^g = \beta_{\text{univ}}(\gamma)$, see also [3, Ex. 6.1].

APPENDIX A. THE ENDOMORPHISM σ_T ON THE SPINOR BUNDLE Σ

Relations (4.1) and (4.3) have another important consequence, related with the eigenspinors of the endomorphism defined by the 4-form σ_T (or equivalently dT) in the Clifford algebra. In particular, the results that we describe below are known (see [31, Lem. 10.7], [11, Thm. 1.1]), but our proofs are different.

Proposition A.1. ([31, Lem. 10.7], [11, Thm. 1.1]) *On a nearly Kähler manifold (M^6, g, J) the spinor fields $\varphi^\pm \in \Sigma_{\pm 2\|T\|}$ are eigenspinors of the endomorphism defined by the 4-form σ_T . In particular,*

$$\sigma_T \cdot \varphi^\pm = -\frac{\text{Scal}^c}{4} \cdot \varphi^\pm = -\frac{3}{2}\|T\|^2 \cdot \varphi^\pm = -3\tau_0 \cdot \varphi^\pm.$$

Similarly, on a proper nearly parallel G_2 -manifold the spinor field $\varphi_0 \in \Sigma_{-\sqrt{7}\|T\|}$ is eigenspinor of the endomorphism defined by the 4-form σ_T . In particular,

$$\sigma_T \cdot \varphi_0 = -\frac{\text{Scal}^c}{4} \cdot \varphi_0 = -3\|T\|^2 \cdot \varphi_0 = -\frac{7}{12}\tau_0^2 \cdot \varphi_0.$$

Proof. We present a direct proof based on (4.1) and (4.3), respectively. Consider a local orthonormal frame $\{e_i\}$. In the nearly Kähler case (M^6, g, J) and due to (4.1), it holds that

$$(e_i \lrcorner T) \cdot \varphi^\pm = \mp \|T\| e_i \cdot \varphi^\pm, \quad \forall i \in \{1, \dots, 6\}.$$

Then, because $(2\sigma_T - 3\|T\|^2) \cdot \varphi^\pm = \sum_i (e_i \lrcorner T) \cdot (e_i \lrcorner T) \cdot \varphi^\pm$ (see Appendix C in [3]), we immediately get

$$\begin{aligned} (2\sigma_T - 3\|T\|^2) \cdot \varphi^\pm &= \mp \|T\| \sum_i (e_i \lrcorner T) \cdot e_i \cdot \varphi^\pm = \mp 3\|T\| T \cdot \varphi^\pm \\ &= \mp 3\|T\| (\pm 2\|T\|) \cdot \varphi^\pm = -6\|T\|^2 \cdot \varphi^\pm. \end{aligned}$$

Similarly, for a 7-dimensional (proper) nearly parallel G_2 -manifold (M^7, g, ω) we easily conclude that

$$\begin{aligned} (2\sigma_T - 3\|T\|^2) \cdot \varphi_0 &= \sum_i (e_i \lrcorner T) \cdot (e_i \lrcorner T) \cdot \varphi_0 \stackrel{(4.3)}{=} \frac{3\|T\|}{\sqrt{7}} \sum_i (e_i \lrcorner T) \cdot e_i \cdot \varphi_0 \\ &= \frac{9\|T\|}{\sqrt{7}} T \cdot \varphi_0 \stackrel{(4.2)}{=} -9\|T\|^2 \cdot \varphi_0, \end{aligned}$$

and the claim follows. \square

Combining the expression for σ_T and the equality $T^2 = -2\sigma_T + \|T\|^2$, it is easy to compute also the action of T^2 on ∇^c -parallel spinors lying in Σ_γ . In particular, for a ∇^c -parallel spinor φ the action $T^2 \cdot \varphi$ is encrypted in the kernel of the Casimir operator $\Omega := \Delta_T + \frac{1}{16} [2\text{Scal}^g + \|T\|^2] - \frac{1}{4}T^2$ [7]. Any ∇^c -parallel belongs in the kernel of Ω , hence it satisfies the equation (see for example [8])

$$T^2 \cdot \varphi = \frac{1}{4} [2\text{Scal}^g + \|T\|^2] \cdot \varphi.$$

This formula in combination with $T^2 = -2\sigma_T + \|T\|^2$, gives rise to another way for the computation of $\sigma_T \cdot \varphi$. Finally, for the action $T^2 \cdot \varphi$ the relation $T \cdot \varphi = \gamma \varphi$ can be applied twice which yields the same result, i.e. $T^2 \cdot \varphi = \gamma^2 \varphi$ with $\gamma^2 = \frac{1}{4} [2\text{Scal}^g + \|T\|^2]$ by Theorem 3.1.

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